New developments on the analytical cascade swaption calibration of the Libor Market Model

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Abstract

In this work we focus on the swaptions "second-order automatic cascade calibration" methodology for the Libor Market Model (LMM), first appeared in Brigo and Mercurio (2001). Such analytical calibration induces a direct correspondence between swaption market volatilities and LMM volatility parameters. We present the basic algorithm and a possible extension, pointing out its main features and some typical problems. The procedure requires an exogenous forward rates instantaneous correlation matrix, so that a historically estimated matrix is considered. This is used to find the parameters of parsimonious parametric forms, in order to make them reflect some key characteristics of the historical matrix. A calibration case study is presented, using different correlations and analyzing their impact on the results. We also consider the consequences of techniques used for problems related to missing or illiquid input data, and analyze calibration results in terms of terminal correlations and evolution of the term structure of volatilities. We further assess, via Monte Carlo simulation of the LMM dynamics,the reliability of the underlying approximation leading to the cascade calibration in this specific context, and finally present some possibilities to include information coming from the cap market.

Contents

1 Introduction

In this work we consider the calibration of the *Libor Market Model* (*LMM*), the most popular model within the family of interest rate models known as *market models*. The LMM owes its popularity to its compatibility with Black's market formula for caps. Although the LMM is not compatible with Black's market formula for swaptions (exactly reproduced, instead, by the *Swap Market Model*), many empirical studies have shown the discrepancy to be practically negligible in most situations, see for instance Brigo and Liinev (2002). Accordingly, an analytical approximation for efficient swaptions pricing in the LMM, well-known among practitioners, is recalled in the following.

This approximation, combined with a particular volatility parameterization, can lead to an automatic, fast and analytical swaptions calibration of the LMM, first appeared in Brigo and Mercurio (2001). Such swaptions "second-order automatic cascade calibration" induces a direct correspondence between swaption market volatilities and LMM volatility parameters and, provided that the above industry approximation is used for pricing, allows to exactly recover the input market swaption prices.

This methodology will be the main focus in the following. Therefore, we begin by quickly recalling some basic facts about the LMM dynamics, its covariance structures, and the approximation for swaption prices, and then we move on to describe the cascade calibration procedure. The basic algorithm, and a possible extension, are presented, pointing out the main advantages of the method and some typical problems. These features provide the motivations for the following developments, as outlined below and in more detail in Section 3.2.

First, we are faced with the issue of choosing a suitable exogenous correlation matrix. This leads us to start with a historical estimation of the correlations among forward rates, then to the analysis of different parsimonious parametric forms with desirable properties, and finally to the examination of possible choices to find parameters reflecting some key characteristics of the historical matrix.

The correlations obtained, at various ranks, are applied to real market calibration cases, examining their impact on the results. This is done in order to avoid a typical flaw of similar cascade algorithms: the occurrence of nonsensical calibrated parameters, such as negative or complex volatilities. Again with reference to the need of avoiding similar results, we also consider the impact of techniques used for problems related to missing or illiquid input data. Several configurations generating acceptable results are found and their features examined. We further assess the regularity of calibration results in terms of terminal correlations and evolution of the term structure of volatilities.

At this point, we want to check the reliability of the approximation leading to the cascade calibration. Consequently, we test the accuracy of such approximation by means of Monte Carlo simulations of the true LMM dynamics, under assumptions on the covariance structures which are typical of the cascade calibration and different, or more general, than in analogous previous tests. Finally, we present and implement some possibilities to include information coming from the cap market, and comment on the results.

2 The Libor Market Model

2.1 Dynamics and volatility structures

Consider a set $\{T_0, \ldots, T_M\}$ of expiry-maturity pairs of dates for a family of spanning forward rates (a set called sometimes in the following *tenor structure*). We shall denote by $\{\tau_0, \ldots, \tau_M\}$ the corresponding year fractions, so that, calling $T_{-1} = 0$ the current time, τ_i is the year fraction associated with the pair of dates T_{i-1}, T_i.

Denote by $F_k(t)$ the generic simply compounded forward rate $F(t; T_{k-1}, T_k)$, $k = 1, \ldots, M$, that is the forward rate resetting at T_{k-1} (expiry) and with maturity T_k. Consider now the T_k-forward adjusted probability measure Q^k associated to the zero coupon bond numeraire with maturity T_k. The Libor Market Model assumes the following (driftless) lognormal dynamics for $F_k(t)$ under Q^k:

$$dF_k(t) = \sigma_k(t) F_k(t)\, dZ_k^k(t), \quad t \leq T_{k-1},$$

where $Z_k^k(t)$ is the k-th component of an M-dimensional Brownian motion $Z^k(t)$ under Q^k, with instantaneous covariance $\rho = (\rho_{i,j})_{i,j=1,\ldots M}$, so that

$$dZ^k(t)\, dZ^k(t)' = \rho\, dt\ .$$

Notice that the upper index in the Brownian motion denotes the measure, while the lower index denotes the vector component. We will often omit the upper index.

The time function $\sigma_k(t)$ bears the usual interpretation of instantaneous volatility at time t for the forward Libor rate F_k. In the following we consider piecewise-constant instantaneous volatilities, thus

$$\sigma_k(t) = \sigma_{k,\beta(t)},$$

where in general $\beta(t) = m$ if $T_{m-2} < t \leq T_{m-1}$, and $\sigma_k(0) = \sigma_{k,1}$. In particular, if we keep the most general of such parameterizations, called *General Piecewise-Constant parameterization* (in short *GPC*), it is possible to organize instantaneous volatilities in a matrix as follows:

Instant. Vols	Time: $t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\ldots$	$(T_{M-2}, T_{M-1}]$
Fwd Rate: $F_1(t)$	$\sigma_{1,1}$	Dead	Dead	$\ldots$	Dead
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	Dead	$\ldots$	Dead
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\ldots$	$\sigma_{M,M}$

TABLE 1: GPC parameterization.

Several assumptions can be made on the entries of Table 1 so as to reduce the number of volatility parameters. Moreover, non piecewise-constant parsimonious parameterizations can be preferred. See Brigo and Mercurio (2001) for several proposals. In the following, we will stick to the GPC formulation, since it is the one leading to the cascade calibration.

By means of the *Change of Numeraire Technique*, we can work out the dynamics of F_k under the forward adjusted measure Q^i in the three cases $i < k$, $i = k$ and $i > k$. They are as follows.

- $i < k, \quad t \leq T_i$:

$$dF_k(t) = \sigma_k(t)F_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}\, dt + \sigma_k(t)F_k(t)\, dZ_k(t), \quad (1)$$

- $i = k, \quad t \leq T_{k-1}$:

$$dF_k(t) = \sigma_k(t)F_k(t)\, dZ_k(t),$$

- $i > k, \quad t \leq T_{k-1}$:

$$dF_k(t) = -\sigma_k(t)F_k(t) \sum_{j=k+1}^{i} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}\, dt + \sigma_k(t)F_k(t)\, dZ_k(t),$$

where $Z(t) = Z^i(t)$ is an (M-dimensional) Brownian motion under Q^i.

The above dynamics describe the Libor Market Model, and do not feature known marginal or transition densities. As a consequence, no analytical formula or simple numeric integration can be used in order to price contingent claims depending on the joint dynamics.

2.2 Swaptions pricing and correlations modeling

As we hinted at in the introduction, the market model designed for swaptions pricing, and hence reproducing Black's market formula for swaptions, is the Swap Market Model (*SMM*). Denoting by $S_{\alpha,\beta}(t)$ a forward swap rate at time t for an interest rate swap with first reset at T_α and exchanging payments at $T_{\alpha+1}, \ldots, T_\beta$, this model assumes a lognormal dynamics for $S_{\alpha,\beta}(t)$ under the related swap measure $Q^{\alpha,\beta}$. Thus

$$dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t)S_{\alpha,\beta}(t)\, dW_t^{\alpha,\beta}, \quad (2)$$

where $W^{\alpha,\beta}$ is a standard Brownian motion under $Q^{\alpha,\beta}$.

If we choose the Libor Market Model as central model, we must resort to different pricing techniques. It is possible to price swaptions with a Monte Carlo simulation, by simulating the forward rates involved in the payoff through a discretization of the dynamics presented above, so as to obtain the relevant zero coupon bonds and the forward swap rate. In fact, recall that the following relationships hold:

$$\begin{aligned} S_{\alpha,\beta}(t) &= \sum\nolimits_{i=\alpha+1}^{\beta} w_i(t)F_i(t), \\ w_i(F_{\alpha+1}(t), \ldots, F_\beta(t)) &= \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}}. \end{aligned} \quad (3)$$

However, analytical approximations are available for swaptions in the LMM. To introduce the formula we will use in the following, consider that a crucial role in the SMM model is played by the Black swap volatility $v_{\alpha,\beta}(T_\alpha)$ entering Black's formula for swaptions, expressed by

$$v^2_{\alpha,\beta}(T_\alpha) = \frac{1}{T_\alpha}\int_0^{T_\alpha} (\sigma^{(\alpha,\beta)}(t))^2 \ dt. \tag{4}$$

One can compute, under a number of approximations, based on "partially freezing the drift" and on "collapsing all measures" in the original dynamics, an analogous quantity $v^{\text{LMM}}_{\alpha,\beta}$ in the LMM. We present here one of the simplest formulas based on a similar setting, appeared earlier for example in Rebonato (1998), and tested against Monte Carlo simulations for instance in Brigo and Mercurio (2001). Such approximated formula, we already alluded to in the introduction, reads:

$$(v^{\text{LMM}}_{\alpha,\beta})^2 = \frac{1}{T_\alpha}\sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2}\int_0^{T_\alpha} \sigma_i(t)\sigma_j(t) \ dt. \tag{5}$$

The quantity $v^{\text{LMM}}_{\alpha,\beta}$ can be used as a proxy for the Black volatility $v_{\alpha,\beta}(T_\alpha)$ of the swap rate $S_{\alpha,\beta}$. Putting this quantity in Black's formula for swaptions allows one to compute approximated swaptions prices with a closed form formula under the Libor Market Model.

Now observe that, contrary to caps, swaption payoffs cannot be decomposed additively in payoffs depending only on single forward rates, so that swaption prices (computed either through Monte Carlo simulation or via analytical approximations) will depend on the correlation ρ, differently from caps. Modeling instantaneous correlations is a delicate task. A viable correlation matrix must be symmetric, positive semidefinite, with all entries in $[-1, 1]$ and diagonal elements equal to 1. In addition, what really matters is choosing a structure flexible enough to express a large number of swaption prices and, at the same time, parsimonious enough to be tractable. For example, we may note that historical one-factor short rate models imply perfectly instantaneously correlated forward rates, i.e. $\rho_{i,j} = 1, \forall i, j$. Notice that the rank of the instantaneous correlation matrix equals the number of stochastic factors entering the model. Compared to this single factor case, one needs to lower the correlation of the forward rates implied by the model, by appropriately modeling instantaneous correlations, and also by carefully redistributing integrated variances of forward rates over time. More generally, we remark that instantaneous correlations of very low rank formulations have a sigmoid-like shape that, initially, cannot decrease quickly. This means that very low rank models are liable to generate too strong a correlation between adjacent forward rates and too a weak one between distant rates, leading to swaptions mispricing, as underlined, for instance, by Rebonato (1999).

On the other hand, a freely calibrated or historically estimated matrix will usually be full rank, that is of rank M, where M is the number of forward rates embedded in the relevant swap rate or, more generally, involved in the model. Therefore it can be definitely high. In case we have to resort to simulation techniques, this is an undesirable feature, in that it implies as many stochastic factors in the model. Moreover, several empirical surveys, see for example Brace, Dun and Barton (1998), show that a reduced rank can be enough for most practical purposes.

According to these observations, we can consider $dZ(t)$ as given by $B\, dW(t)$, with W an n-dimensional standard Brownian motion, $n \le M$, and $B = (b_{i,j})$ a suitable n-rank $M \times n$ matrix. In this way $\rho^B = BB'$ is the n-rank correlation matrix, and consequently we can reduce the noise factors. We follow Rebonato and Jäckel (1999) by taking $M(n-1)$ angles θ to parameterize B:

$$\begin{aligned} b_{i,1} &= \cos\theta_{i,1} \\ b_{i,k} &= \cos\theta_{i,k}\sin\theta_{i,1}\cdots\sin\theta_{i,k-1}, \quad 1<k<n, \\ b_{i,n} &= \sin\theta_{i,1}\cdots\sin\theta_{i,n-1}. \end{aligned}$$

The other purpose we would like to attain is a reduction in the number of parameters, compared to the original $M(M-1)/2$ different entries of a correlation matrix. This can be desirable for ease of computation, but also for different reasons, such as those mentioned in Section 4. By means of this parameterization, such a goal can be achieved only if we choose a low n, in particular such that $(n-1) < (M-1)/2$. This solution can be undesirable due to the above "low-rank pattern" problems. On the other hand, if one keeps high rank there will be even an increase in the parameter space dimension. For example, as highlighted by Rapisarda, Brigo and Mercurio (2002), keeping rank M we go from the original $M(M-1)/2$ to $M(M-1)$ parameters. In that work, the authors propose a way to eliminate this redundancy, based on an appealing geometrical interpretation. However, if we are interested in a reduction in comparison with the number of different entries in ρ, we have to resort to other approaches, some of which will be presented later on in Section 4.1.

2.3 Term structure of volatility and terminal correlation

The *term structure of volatility* (sometimes *TSV* in the following) at time T_j is a graph of expiry times T_{h-1} against average volatilities $V(T_j, T_{h-1})$ of the forward rates $F_h(t)$ up to that expiry time itself, i.e. for $t \in (T_j, T_{h-1})$. In other terms, at time $t = T_j$, the volatility term structure is the graph of points

$$\{(T_{j+1}, V(T_j, T_{j+1})), (T_{j+2}, V(T_j, T_{j+2})), \dots, (T_{M-1}, V(T_j, T_{M-1}))\},$$

where

$$V^2(T_j, T_{h-1}) = \frac{1}{T_{h-1}-T_j}\int_{T_j}^{T_{h-1}} \frac{dF_h(t)\, dF_h(t)}{F_h(t)F_h(t)} = \frac{1}{T_{h-1}-T_j}\int_{T_j}^{T_{h-1}} \sigma_h^2(t)\, dt$$

for $h > j+1$. The term structure of volatilities at time 0 is given simply by caplet volatilities plotted against their expires (reminding that caplets are unitary caps).

Different assumptions on the behaviour of instantaneous volatilities imply different evolutions for the term structure of volatilities in time. Such evolution is deterministic in the LMM and can be displayed once the model has been calibrated. So it is often examined in order better to understand the implications of a particular parameterization of the instantaneous volatility. In particular, one would like a calibration of the LMM to feature a generally realistic, smooth and qualitatively stable evolution of the term structure.

A similar use can be made of another quantity of interest, the so called *terminal correlation* (that at times we will denote by TC). In general, assume one is interested in computing the terminal correlation, as implied by the LMM, between the forward rates F_i and F_j at a future time instant T_α, $\alpha \leq i-1 < j$, say under the measure Q^γ, $\gamma \geq \alpha$. Then one needs to compute

$$\mathrm{Corr}^\gamma(F_i(T_\alpha), F_j(T_\alpha)) =$$
$$\frac{E^\gamma\left[(F_i(T_\alpha) - E^\gamma F_i(T_\alpha))(F_j(T_\alpha) - E^\gamma F_j(T_\alpha))\right]}{\sqrt{E^\gamma[(F_i(T_\alpha) - E^\gamma F_i(T_\alpha))^2]}\sqrt{E^\gamma[(F_j(T_\alpha) - E^\gamma F_j(T_\alpha))^2]}}$$

Recalling the dynamics of F_i and F_j under Q^γ in the LMM, the expected values appearing in the above expression can be obtained by a Monte Carlo simulation of F_i and F_j up to time T_α. However, at times traders may need to quickly check reliability of the model's terminal correlations, so that there could be no time to run a Monte Carlo simulation. Fortunately, there do exist approximated formulas that allow us to compute terminal correlations algebraically from the LMM parameters ρ and σ. The first approximation we introduce is a partial freezing of the drift in the dynamics, and a collapse of all forward measures. Following Brigo and Mercurio (2001), we obtain easily

$$\mathrm{Corr}(F_i(T_\alpha), F_j(T_\alpha)) = \frac{\exp\left(\int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\rho_{i,j}\, dt\right) - 1}{\sqrt{\exp\left(\int_0^{T_\alpha} \sigma_i^2(t)\, dt\right) - 1}\,\sqrt{\exp\left(\int_0^{T_\alpha} \sigma_j^2(t)\, dt\right) - 1}}. \tag{6}$$

A first order expansion of the exponentials appearing in this formula yields Rebonato's (1999) terminal correlation formula, enjoying particularly interesting properties. In facts, it leads to *terminal correlations always smaller, in absolute value, than instantaneous correlations*, and through a careful repartition of integrated volatilities (caplets) in instantaneous volatilities we can make such terminal correlation arbitrarily close to zero, even when the instantaneous correlation $\rho_{i,j}$ is one. See Brigo and Mercurio (2001) for more details, examples, and for numerical tests against Monte Carlo showing that both approximations are good in non-pathological situations. In the following, when examining market calibration cases, we will make use of formula (6) to calculate the terminal correlation matrix associated with calibrated parameters, checking if it is regular and consistent with the typical properties of a forward rates correlation matrix, that will be presented in Section 4.1. These are the same characteristics considered desirable also for the instantaneous correlations. Therefore, when instantaneous correlations already enjoy similar properties, we would appreciate them not to be radically spoiled when moving to terminal correlations.

3 Quick and analytical swaptions cascade calibration

In general, swaptions calibration is a difficult and time-consuming task within the Libor Market Model. In fact, whereas caps are the natural reference market for the model, so that market caplet volatilities can be recovered almost automatically, this is not the case for swaptions. Calibrating the model to swaptions means searching for those values of the parameters that make the model swaption prices as close as possible to the market prices. Therefore calibration usually involves a minimization of some loss function expressing the distance between model and market prices. Furthermore, swaptions in the LMM should be in principle priced via Monte Carlo simulation. In this way, the prices used in the optimization would be the output of simulations, leading to a computationally burdensome procedure. Analytical approximations can avoid the simulation, but usually not the minimization routine. A further consequence is that in general input market prices will not be exactly matched, in that there will be a calibration error, however minimized it may be.

Surprisingly enough, the most general piecewise-constant parameterization, the GPC of Table 1, associated with the industry approximated formula (5), makes it possible to find a calibration procedure not requiring either simulation or optimization. This quick and analytical calibration methodology, presented in Brigo and Mercurio (2001) and (2002), in fact reduces itself to the solution of a cascade of second-order algebraic equations, provided that the correlation matrix is given exogenously. Besides, the calibration is based on the possibility to invert formula (5), so that, if this approximation is used for pricing, then input market prices are exactly retrieved, without calibration error. Finally, a one-to-one correspondence between σ's and market volatilities is introduced, at least in the basic algorithm.

In order to outline the derivation, we first present how formula (5) reads in case of GPC parameterization:

$$(v_{\alpha,\beta})^2 \approx \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{T_\alpha \; S_{\alpha,\beta}(0)^2} \sum_{h=0}^{\alpha}(T_h - T_{h-1})\sigma_{i,h+1}\sigma_{j,h+1}. \tag{7}$$

Keep in mind that the weights w are specific of the swaption being considered, i.e. they depend on α and β. Secondly, we need to see how the market organizes swaption prices in a table. Traders typically consider a matrix of at-the-money (ATM) swaptions Black's volatilities organized as follows. To simplify ideas, assume we are interested only in swaptions with maturity and underlying swap length given by multiples of one year. Each row is indexed by the swaption maturity T_α, whereas each column is indexed in terms of the underlying swap length, $T_\beta - T_\alpha$. The $x \times y$-swaption is then the swaption in the table whose maturity is x-years and whose underlying swap is y years long. A typical example of table of swaption volatilities is shown below:

	1	2	3	4	5	6	7	8	9	10
1	**17.90**	**16.50**	**15.30**	**14.40**	**13.70**	**13.20**	**12.80**	**12.50**	**12.30**	**12.00**
2	**15.40**	**14.20**	**13.60**	**13.00**	**12.60**	**12.20**	**12.00**	**11.70**	**11.50**	11.30
3	**14.30**	**13.30**	**12.70**	**12.20**	**11.90**	**11.70**	**11.50**	**11.30**	11.10	10.90
4	**13.60**	**12.70**	**12.10**	**11.70**	**11.40**	**11.30**	**11.10**	10.90	10.80	10.70
5	**12.90**	**12.10**	**11.70**	**11.30**	**11.10**	**10.90**	10.80	10.60	10.50	10.40
6	***12.50***	***11.80***	***11.40***	***10.95***	***10.75***	*10.60*	*10.50*	*10.40*	*10.35*	*10.25*
7	**12.10**	**11.50**	**11.10**	**10.60**	10.40	10.30	10.20	10.20	10.20	10.10
8	***11.80***	***11.20***	***10.83***	*10.40*	*10.23*	*10.17*	*10.10*	*10.10*	*10.07*	*10.00*
9	***11.50***	***10.90***	*10.57*	*10.20*	*10.07*	*10.03*	*10.00*	*10.00*	*9.93*	*9.90*
10	**11.20**	10.60	10.30	10.00	9.90	9.90	9.90	9.90	9.80	9.80

TABLE 2: Black implied volatilities of ATM swaptions on February 1, 2002.

We remark that the rows associated with the swaptions maturities of $6, 8$ and 9 years have been written in italics. The reason is that they do not refer to market quotations, since the corresponding maturities are not quoted on the Euro market. Considering that this methodology requires a complete swaption matrix, featuring values for each and every maturity (and length) in the range, they have been obtained by a simple linear interpolation between the adjacent market values on the same columns, as typical in the literature, see for instance Brigo and Mercurio (2001) and Joshi and Rebonato (2001). The possible implications of such interpolation will be considered in Section 5.2. In addition, keep in mind that a similar matrix is not necessarily updated uniformly: while the most liquid swaptions are updated regularly, some entries may refer to older market situations. The problem of the inconsistency due to this "temporal misalignment", and its possible effects, will be addressed later on. Finally, notice that the values in the upper triangular part, with reference to the secondary diagonal included, are in bold. This is to underline that, as we will se below, the basic algorithm applies only to these values, while further hypotheses will be needed to extend it to the entire swaption matrix. Then the entries we will consider in the basic algorithm are those such that, denoting by $(i + j)$ the sum of the column and row indices and by s the matrix dimension, the following condition holds

$$(i + j) \leq (s + 1)\,,$$

so that we refer to swaption volatilities $v_{\alpha,\beta}$ for which $\beta \leq s$.

3.1 The basic algorithm and a possible extension

We start from the swaption volatilities in the upper half of the swaption matrix (the part in bold) and move along the table. Let us start from the $(1, 1)$ entry $v_{0,1}$. Use the approximated Formula (7) and compute, after straightforward simplifications, $(v_{0,1})^2 \approx \sigma_{1,1}^2$. This formula is immediately invertible and provides us with the volatility parameter $\sigma_{1,1}$ as a function of the swaption volatility $v_{0,1}$. Now move on to the right, to entry $(1, 2)$, containing $v_{0,2}$. The same formula gives, this time,

$$\begin{aligned} S_{0,2}(0)^2(v_{0,2})^2 \quad &\approx \quad w_1(0)^2 F_1(0)^2\sigma_{1,1}^2 + w_2(0)^2 F_2(0)^2\sigma_{2,1}^2 \\ &\qquad +2\rho_{1,2} w_1(0)F_1(0)w_2(0)F_2(0)\sigma_{1,1}\sigma_{2,1}\ . \end{aligned}$$

Everything in this formula is known, except $\sigma_{2,1}$. We then solve the elementary-school algebraic second order equation in $\sigma_{2,1}$, and recover analytically $\sigma_{2,1}$ in terms of the previously found $\sigma_{1,1}$ and of the known swaptions data. Now move on to the right again, to entry $(1, 3)$, containing $v_{0,3}$. In this case $\sigma_{3,1}$ turns out to be the only unknown, and so forth. If we go on moving left to right within the upper part of the table until we reach the end of the row (namely an entry on the secondary diagonal), and then moving top down to the beginning of the next row, each time only one new σ appears, and this makes the relationship between the v's and the σ's analytically invertible. The following table summarizes, for a 3×3 matrix, the dependence of the swaptions volatilities v from the instantaneous forward volatilities σ, as far as the upper part of the swaption matrix is concerned.

Length / Maturity	1y	2y	3y
$T_0 = 1$y	$v_{0,1}$ $\sigma_{1,1}$	$v_{0,2}$ $\sigma_{1,1}, \sigma_{2,1}$	$v_{0,3}$ $\sigma_{1,1}, \sigma_{2,1}, \sigma_{3,1}$
$T_1 = 2$y	$v_{1,2}$ $\sigma_{2,1}, \sigma_{2,2}$	$v_{1,3}$ $\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \sigma_{3,2}$	- -
$T_2 = 3$y	$v_{2,3}$ $\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}$	-	-

In reading the table left to right and top down, one realizes that indeed each time we consider a new swaption volatility $v^2_{\alpha,\beta}$ only one new σ appears, that is $\sigma_{\beta,\alpha+1}$. More generally, we can notice that $v^2_{\alpha,\beta}$ depends on all $\sigma_{k,\varepsilon}$ such that

$$\begin{cases} \alpha < k \leq \beta \\ \varepsilon \leq (\alpha+1) \end{cases} . \tag{8}$$

We can now give the general method for calibrating our volatility formulation of Table 1 to the upper triangle of the swaption matrix when an arbitrary number s of rows of the matrix is given. First, rewrite Formula (7) as follows:

$$\begin{aligned} & T_\alpha \; S_{\alpha,\beta}(0)^2 \; v^2_{\alpha,\beta} = \\ & \sum_{i,j=\alpha+1}^{\beta-1} w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j} \; \sum_{h=0}^{\alpha}(T_h - T_{h-1}) \; \sigma_{i,h+1} \; \sigma_{j,h+1} \\ + \quad & 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta(0)w_j(0)F_\beta(0)F_j(0)\rho_{\beta,j} \; \sum_{h=0}^{\alpha-1}(T_h - T_{h-1}) \; \sigma_{\beta,h+1} \; \sigma_{j,h+1} \\ + \quad & 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta(0)w_j(0)F_\beta(0)F_j(0)\rho_{\beta,j} \; (T_\alpha - T_{\alpha-1}) \; \boxed{\sigma_{\beta,\alpha+1}} \; \sigma_{j,\alpha+1} \\ + \quad & w_\beta(0)^2 F_\beta(0)^2 \sum_{h=0}^{\alpha-1}(T_h - T_{h-1}) \; \sigma^2_{\beta,h+1} \\ + \quad & w_\beta(0)^2 F_\beta(0)^2 (T_\alpha - T_{\alpha-1}) \; \boxed{\sigma^2_{\beta,\alpha+1}} . \end{aligned} \tag{9}$$

In turn, by suitable definition of the coefficients A, B and C this equation can be rewritten as:

$$A_{\alpha,\beta}\sigma^2_{\beta,\alpha+1} + B_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C_{\alpha,\beta} = 0$$

and thus it can be solved analytically by the usual elementary formula. Moreover, if $A_{\alpha,\beta}$ and $B_{\alpha,\beta}$ are strictly positive (as they both are when correlations are, as usual, positive) its smaller solution, when real, is always negative. In this case we take therefore the greater solution, which, if real and positive, is just the volatility $\sigma_{\beta,\alpha+1}$ we are looking for. The solution is not always positive or even real, as will be seen later on.

Now that we have written down the general structure of the equations, we can draw up a scheme for the calibration algorithm:

1. Select the number s of rows in the swaption matrix that are of interest for the calibration;

2. Set $\alpha = 0$;

3. Set $\beta = \alpha + 1$;

4. Solve equation (9) in $\sigma_{\beta,\alpha+1}$.

5. Increase β by one. If β is smaller than or equal to s, go back to point 4, otherwise increase α by one.

6. If $\alpha < s$ go back to point 3, otherwise stop.

It is important to point out once more that, if one follows this algorithm, i.e. if the upper half of the swaption matrix is visited from left to right and top down, then when one comes to point 4 all quantities in the equation to be solved are indeed known with the exception of our unknown $\sigma_{\beta,\alpha+1}$. Since the algorithm consists of a cascade of second order equations, this method is at times referred to as *cascade calibration*.

Notice that in this case results are independent of s, in that the output of the calibration to a sub-matrix will be a subset of the output of a calibration to the original matrix. This implies also that any swaption matrix V can be seen as a sub-matrix of a larger one including V in its upper triangular part, so that all entries of V, including those in the lower triangle, will be recovered by applying this algorithm to the upper part of the larger matrix. In other words, this "nested consistency" means that, if all needed market values were available, this upper part algorithm might be considered general, with no need for any extension. Of course this is not usually the case, so that we may wonder about what happens when in need of recovering the whole matrix. If we apply the above algorithm extending it to the elements in the lower part, namely we keep on moving from left to right and top down but now visiting all the boxes in the matrix, we are faced with a problem. In fact, in certain positions of the table we will have more than one unknown in formula (9). However, we can still manage by assuming these unknowns to be related in some way. For example, following Brigo e Mercurio (2001), we can assume them to be equal to each other. Then we can still solve the equation analytically in a single unknown.

Let us detail this point a little more. Suppose that we visit the rectangular swaption matrix from left to right and top down. Suppose further that we apply the above algorithm each time we have a single unknown, and that we determine all the multiple unknowns by inverting the usual formula as soon as they occur, for instance assuming them to be equal. In this case, it is easy to see that the multiple-unknowns situation will occur only when reaching swaptions volatilities in the last column of the matrix (with the exception, of course, of the first-row element of such column, still belonging to the upper part of the swaption matrix). Some more details are given in Morini (2002).

Taking these observations into account, we can easily generalize the above "upper triangular" algorithm to the entire swaption matrix. Let us start with point 5. The related condition is no longer $\beta \leq s$, but $(\beta - \alpha) \leq s$. Furthermore, in case $\beta = s + \alpha$, i.e. when one reaches one entry on the last column (with the exception, as already said, of the very first one), the new point 4 requires in this case to assume all the unknowns to be equal to the standard unknown $\sigma_{\beta,\alpha+1}$, i.e. we assume

$$\sigma_{\beta,\alpha+1} = \sigma_{\beta,\alpha} = \ldots = \sigma_{\beta,1} \quad \text{for} \quad \beta = s + \alpha.$$

Hence the new equation to solve is

$$A^*_{\alpha,\beta}\sigma^2_{\beta,\alpha+1} + B^*_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C^*_{\alpha,\beta} = 0,$$

where

$$A^*_{\alpha,\beta} = w_\beta(0)^2 F_\beta(0)^2 (T_\alpha - T_{\alpha-1}) + w_\beta(0)^2 F_\beta(0)^2 \sum_{h=0}^{\alpha-1} (T_h - T_{h-1}),$$

$$\begin{aligned} B^*_{\alpha,\beta} \;\; = \;\; & 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta(0) w_j(0) F_\beta(0) F_j(0) \rho_{\beta,j} \; (T_\alpha - T_{\alpha-1}) \; \sigma_{j,\alpha+1} \\ & +2 \sum_{j=\alpha+1}^{\beta-1} w_\beta(0) w_j(0) F_\beta(0) F_j(0) \rho_{\beta,j} \; \sum_{h=0}^{\alpha-1} (T_h - T_{h-1}) \; \sigma_{j,h+1}, \end{aligned}$$

$$\begin{aligned} C^*_{\alpha,\beta} \;\; = \;\; & \sum_{i,j=\alpha+1}^{\beta-1} w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j} \; \sum_{h=0}^{\alpha} (T_h - T_{h-1}) \; \sigma_{i,h+1} \; \sigma_{j,h+1} \\ & -T_\alpha \; S_{\alpha,\beta}(0)^2 \; v^2_{\alpha,\beta}. \end{aligned}$$

The rest of the algorithm keeps unchanged. Now that we have presented the method, let us see in more detail its main advantages and typical problems.

3.2 Cascade calibration: opportunities and problems

First, it is worthwhile summarizing the main features of this calibration method:

1. *The correlation matrix is an exogenous input*;
2. The remaining inputs are *a complete swaption volatilities matrix and the zero coupon curve, so cap data are not involved in the calibration*;
3. *The calibration can be carried out through closed form formulas*;
4. If industry formula (5) is used for pricing swaptions in combination with Black's formula, *market swaption prices are recovered exactly*;
5. The method establishes *a one-to-one correspondence between model volatility parameters and market swaption volatilities*, at least in its basic form.

The last three points clearly represent the main advantages of this calibration methodology with respect to traditional LMM swaptions or joint calibration. And the first point too represents an opportunity, in that it allows for imposing satisfactory instantaneous correlations. At the same time, all these characteristics lead to important implications when it comes to the application, providing us with the motivations for the following developments, here outlined. First of all, the exogenous correlation matrix must be chosen suitably, an issue we address in Section 4, following the lines already mentioned in the introduction. Different choices to compute correlations are available: we can use a previously calibrated matrix, such as in Brigo and Mercurio (2001), or else we can historically estimate it. We may also choose a synthetic, standardized form with desirable properties, that is to say reflecting typical regularities found in the market and/or designed to obtain acceptable calibration results.

One may wonder what we mean by "acceptable calibration results". This point leads naturally to consider what lies behind both the main advantage and the main drawback of a similar algorithm: the fact of being totally analytical and not requiring any optimization routine. In fact, this leads to an algorithm which is faster and exact, but there is also the other side of the coin. Avoiding any optimization routine, this methodology does not allow to set any constraint on the output, so that there is no guarantee that the calibrated instantaneous volatilities will be real and non-negative, as any viable volatility should be.

On the contrary, earlier tests, see Brigo and Mercurio (2001) and (2002), and Brigo, Mercurio and Morini (2002), present many negative entries in the output, some even depending only on the calibration to the upper part of the swaption matrix. This problem is there overcome by a rather drastic smoothing of the input swaption matrix. Such a method permits to eliminate those weird output volatilities, probably by discarding the influence of misaligned market quotations corresponding to illiquid swaptions. On the other hand, this solution brings about not negligible differences between market original volatilities and smoothed volatilities, some of which can be considered too large for practical purposes, as pointed out by Brigo and Mercurio (2002).

Here we try and find different, less drastic ways to get rid of such inconveniences, as we already alluded to in the introduction. Initially, in Section 5.1, we will test the implications of different correlation matrices. Although it is easy to build toy swaption matrices giving problems for any choice of correlations, with market data we find that they can be decisive, and we detect empirical links between some major characteristics of the correlation matrix, influenced also by the rank, and the occurrence of anomalous outputs. Then in Section 5.2, we address the issue of the relevance of the techniques used for missing or illiquid data. We notice that negative or complex entries are all among those depending also on artificial input swaption volatilities calculated by the simple linear interpolation we hinted at above. Then we test the effects of a different interpolation, based on a seemingly more realistic fitting form that we apply first to missing data, and later also to those entries deemed to be misaligned.

At this point, even though we manage to detect typical configurations generating acceptable results, a few further questions about the reliability and effectiveness of the cascade calibration are still open. The first regards the qualitative properties of the calibrated parameters, that in Section 5.3 we assess by examining, for some calibration cases, the resulting terminal correlations and evolution of the term structure of volatilities. The second, fundamental one concerns the reliability of the approximated industry formula (5). This formula will be tested in Section 6 via Monte Carlo simulation of the LMM dynamics, in the typical context of the cascade calibration, namely GPC parameterization of instantaneous volatilities and various correlation matrices with various ranks. In fact these latter correlation structures are particularly easy to implement in this approach that allows for an exogenous correlation matrix.

As explicitly recalled at point 3 at the beginning of this section, we are not talking about a joint calibration, in that one half of the interest rate derivatives world, the cap market, is left out. However, one might be interested in taking into account the information coming from the cap market. In Section 7 we see some possible choices.

Finally, we remark that further investigations, not considered here, can be carried out to assess, or even improve, the effectiveness of this calibration. For instance, alternative sub-parameterizations can be attempted to avoid the multiple unknowns when extending the algorithm to the entire swaption matrix. One might prefer hypothesis reflecting typical patterns detected in the market, or devised for avoiding negative or complex volatilities. Moreover, the stability over time of the calibrated parameters might be checked by a historical analysis of the fitted volatilities, as suggested by Brigo and Mercurio (2002). In the same paper, also a possible alternative use of cascade calibration is proposed: an analytical calibration as the one here presented can be used, once negative and complex parameters are set to zero, as an initial guess for a constrained optimization. Yet, in the following we will not make use of such an opportunity, since we aim to find directly acceptable results without resorting to traditional optimizations.

4 The exogenous correlation matrix

The exogenous nature of the correlation matrix can be considered an interesting opportunity to introduce in the calibration correlation structures bearing some resemblance to real market patterns, such as those identifiable through historical analysis. On the other hand, there are several reasons which can make undesirable the possibility of plugging historical estimates directly into the calibration routine. First of all, it is well known that European swaptions turn out to be relatively insensitive to correlation details, so that simple, standardized correlations typically produce prices very little different from those obtained using more complex and realistic correlation functions, as pointed out for instance by Jäckel and Rebonato (2000). In case of cascade calibration, this kind of insensitivity is even stronger, in that the calibrated σ's ensure market prices to be exactly matched no matter which instantaneous correlation matrix is imposed exogenously.

Besides, further arguments are leading our interest towards parsimonious forms enjoying desirable properties. First of all, they guarantee a smooth and regular behaviour of correlations, generally better for obtaining regular σ's, whereas historical estimations are usually characterized by problems such as outliers, non-synchronous data and discontinuities in correlation surfaces. These and further problems are recalled by Rebonato e Jäckel (1999), that consequently propose to fit a parametric form onto the estimate. Secondly, parametric forms such those presented in Section 4.1 enjoy particularly interesting properties typical of forward rates correlations. The two points just mentioned acquire further importance when considering that we are also looking for regular terminal correlations with similar properties.

Even more importantly, such forms depend on a low number of parameters, so that we can more easily control the main features of the matrix, detecting those that provoke undesirable anomalous outputs so as to avoid them. The opportunity to easily modify correlations, clearly unavailable when plugging a fixed matrix, is fundamental also for the common market use of correlations. In fact, market operators need often to modify correlations, to incorporate personal views or recent changes in the market, to carry out scenario analysis, and for risk management and hedging purposes. In this event, correlations which are parsimonious in terms of number of parameters and designed to remain regular and well-defined as a whole are of great help.

However, to avoid arbitrariness, we still look for a correlation structure bearing at least some resemblance to the real market situation. Therefore we obtain the correlation matrix in accordance with the following scheme.

1. A market historical correlation matrix is estimated;

2. The parameters of a parsimonious parametric form are determined by keeping the historical estimate as a point of reference;

3. An angles form of the desired rank is fitted to the resulting parsimonious matrix;

Although in this work we have been naturally led to this simple and intuitive procedure, different approaches, not addressed here, might be considered to achieve some of the above targets. For instance, one might think of bayesian estimations incorporating market operators views.

4.1 Parametric forms for correlation

As we said at the end of Section 2.2, we will see here a few parametric forms for correlations permitting a substantial reduction in the number of parameters. In doing so, we also present the typical properties one might desire a forward rates correlation matrix to enjoy.

Some proposals feature a number of parameters as low as M. As far as angles parameterizations are concerned, see for instance the simple sub-parameterization mentioned in Brigo (2002) or the alternative angles setting presented in Rapisarda, Brigo and Mercurio (2002). Switching to a different kind of structures, an M-parameters form by Schoenmakers and Coffey (2000) is based on a finite sequence of positive real numbers

$$1 = c_1 < c_2 < \ldots < c_M, \quad \frac{c_1}{c_2} < \frac{c_2}{c_3} < \ldots < \frac{c_{M-1}}{c_M},$$

assuming

$$\rho^F(c)_{i,j} := c_i/c_j, \quad i \leq j, \quad i,j = 1,\ldots,M.$$

Such a general form can be made more parsimonious by sub-parameterizing the sequence, obtaining for instance a three-parameters form, given by

$$\begin{aligned}\rho_{i,j} = \exp[-|i-j|(\beta - &\frac{\alpha_2}{6M-18}\left(i^2+j^2+ij-6i-6j-3M^2+15M-7\right) \\ +&\frac{\alpha_1}{6M-18}\left(i^2+j^2+ij-3Mi-3Mj+3i+3j+3M^2-6M+2\right)].\end{aligned} \tag{10}$$

where the parameters should be constrained to be non-negative, if one wants to be sure all the typical properties shown below are indeed present. In order to get parameter stability, Schoenmakers and Coffey introduce a change of variables. Subsequently, their calibration experiments point out that one parameter is practically always close to zero. Setting thus it to 0 (equivalent to $\alpha_2 = 0$) leads to the following two-parameter form

$$\begin{aligned}\rho_{i,j} = \exp\Bigg[& - \frac{|i-j|}{M-1}\Bigg(- \ln \rho_\infty \\ & + \eta \frac{i^2 + j^2 + ij - 3Mi - 3Mj + 3i + 3j + 2M^2 - M - 4}{(M-2)(M-3)}\Bigg)\Bigg]. \end{aligned} \tag{11}$$

where $\rho_\infty = \rho_{1,M}$, namely the correlation between the farthest rates in the family, and the following constraint holds: $0 < \eta < -\ln \rho_\infty$.

Both the general form and the versions with fewer parameters enjoy some very interesting properties. Firstly, they are automatically positive semidefinite, as every correlation matrix is to be. Secondly, they produce correlations decreasing as the distance between rates increases. This monotonic *decorrelation* pattern is a typical characteristic of the interest rate market, reproduced also by the classic, two-parameters exponential form given by

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp[-\beta|i-j|], \quad \beta \geq 0. \tag{12}$$

Moreover, the Schoenmakers and Coffey parameterizations also satisfy another desirable and intuitive requirement: the sub-diagonals of the resulting correlation matrix are increasing when moving south-eastwards. This property means that all changes between equally spaced forward rates become more correlated as their expiries increase. In order to gain such a feature also for the exponential classic form above, this must be somehow perturbed, for example as proposed by Rebonato (1998), who sets

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp[-|i-j|(\beta - \alpha\,(\max(i,j)-1))], \tag{13}$$

However, this form is not automatically positive semidefinite, so that one has to check a posteriori that the resulting matrix enjoys such property.

The (full rank) parameterizations we just presented can be used also in conjunction with different parameterizations allowing for a rank reduction, as we hinted at in Section 4. That is what we do in the following calibration examples, when in need of reducing the rank of one of these forms.

4.2 Historical estimation

Considering the use of historical estimation involved in our procedure, outlined at the end of Section 4, we are not interested in going into the details of a complete econometric analysis. That would be beyond the purpose, and the scope, of this work, mainly devoted to cross-section calibration. However, we outline here the main features of the historical estimate used in the following. In estimating correlations, we take into account the particular nature of forward rates in the LMM, characterized by a fixed maturity, contrary to market quotations, where a fixed time-to-maturity is usually considered as time passes. In other words, we observe arrays of discount factors of the following kind

$$P(t, t+Z), P(t+1, t+1+Z), \ldots, P(t+n, t+n+Z),$$

where Z is time-to-maturity, ranging in a standard set of times, whereas what we need are sequences such as

$$P(t,T), P(t+1,T), \dots, P(t+n,T),$$

for the maturities T included in the tenor structure of the chosen LMM. Accordingly, an interpolation between discount factors has been carried out (for the sake of precision, log-interpolation), and only one year of data has been used, since the first forward rate in the family expires in one year from the starting date. These data span the year from February 1, 2001 to February 1, 2002, being the latter the day the data on swaption market volatilities we will use in Section 5 refer to. From these daily quotations of notional zero-coupon bonds, whose maturities range from one to twenty years from today, we extracted daily log-returns of the annual forward rates involved in the model. Starting from the following usual gaussian approximation

$$\left[\ln\left(\frac{F_1(t+\Delta t)}{F_1(t)}\right), ..., \ln\left(\frac{F_{19}(t+\Delta t)}{F_{19}(t)}\right)\right] \sim MN(\mu, V),$$

where $\Delta t = 1$ day, our estimations of the parameters are based on sample mean and covariance for gaussian variables, and are given by

$$\begin{aligned} \hat{\mu}_i &= \frac{1}{m}\sum_{k=0}^{m-1} \ln\left(\frac{F_i(t_{k+1})}{F_i(t_k)}\right), \\ \hat{V}_{i,j} &= \frac{1}{m}\sum_{k=0}^{m-1} \left[\left(\ln\left(\frac{F_i(t_{k+1})}{F_i(t_k)}\right) - \hat{\mu}_i\right)\left(\ln\left(\frac{F_j(t_{k+1})}{F_j(t_k)}\right) - \hat{\mu}_j\right)\right], \end{aligned}$$

where m is the number of days of observation for each rate, so that our estimation of the general correlation element $\rho_{i,j}$ is

$$\hat{\rho}_{i,j} = \frac{\hat{V}_{i,j}}{\sqrt{\hat{V}_{i,i}}\sqrt{\hat{V}_{j,j}}}.$$

The results obtained following the procedure just outlined are shown in Table 3.

Examining the matrix, what is clearly visible is a pronounced and approximately monotonic decorrelation along the columns, when moving away from the diagonal. Particularly noteworthy are the initial steepness of the decorrelation and its proportions. Conversely, the second tendency considered typical of correlation amongst rates, that is the upward trend along the sub-diagonal, is definitely not remarkable. That might be due to the smaller extent of such a phenomenon, more likely to be hidden by noise or differences in liquidity amongst longer rates. Not very different features are visible also in the previous similar estimate showed in Brace, Gatarek and Musiela (1997).

We did some tests on the stability of the estimates, finding out that the values remain rather constant even if we change the sample size or its time positioning. The other kind of analysis we carried out on these data is a simple principal component analysis, revealing that as many as 7 factors are required to explain 90% of the overall variability. This is a far larger number than in most other

	1	2	3	4	5	6	7	8	9	10
1	1.000	0.823	0.693	0.652	0.584	0.467	0.290	0.235	0.434	0.473
2	0.823	1.000	0.798	0.730	0.682	0.546	0.447	0.398	0.529	0.566
3	0.693	0.798	1.000	0.764	0.722	0.629	0.472	0.557	0.671	0.610
4	0.652	0.730	0.764	1.000	0.777	0.674	0.577	0.561	0.681	0.701
5	0.584	0.682	0.722	0.777	1.000	0.842	0.661	0.667	0.711	0.734
6	0.467	0.546	0.629	0.674	0.842	1.000	0.774	0.682	0.729	0.688
7	0.290	0.447	0.472	0.577	0.661	0.774	1.000	0.718	0.709	0.647
8	0.235	0.398	0.557	0.561	0.667	0.682	0.718	1.000	0.735	0.659
9	0.434	0.529	0.671	0.681	0.711	0.729	0.709	0.735	1.000	0.748
10	0.473	0.566	0.610	0.701	0.734	0.688	0.647	0.659	0.748	1.000
11	0.331	0.418	0.484	0.562	0.696	0.770	0.648	0.639	0.591	0.632
12	0.432	0.453	0.519	0.593	0.669	0.694	0.619	0.561	0.665	0.675
13	0.288	0.476	0.483	0.581	0.640	0.659	0.714	0.610	0.688	0.704
14	0.230	0.343	0.542	0.498	0.590	0.634	0.619	0.720	0.693	0.634
15	0.259	0.346	0.462	0.499	0.581	0.615	0.628	0.588	0.690	0.636
16	0.206	0.321	0.422	0.478	0.649	0.677	0.663	0.645	0.634	0.651
17	0.227	0.323	0.450	0.488	0.653	0.702	0.638	0.642	0.644	0.625
18	0.293	0.312	0.420	0.439	0.534	0.569	0.524	0.492	0.518	0.524
19	0.245	0.322	0.352	0.354	0.422	0.447	0.375	0.459	0.402	0.399

	11	12	13	14	15	16	17	18	19
1	0.331	0.432	0.288	0.230	0.259	0.206	0.227	0.293	0.245
2	0.418	0.453	0.476	0.343	0.346	0.321	0.323	0.312	0.322
3	0.484	0.519	0.483	0.542	0.462	0.422	0.450	0.420	0.352
4	0.562	0.593	0.581	0.498	0.499	0.478	0.488	0.439	0.354
5	0.696	0.669	0.640	0.590	0.581	0.649	0.653	0.534	0.422
6	0.770	0.694	0.659	0.634	0.615	0.677	0.702	0.569	0.447
7	0.648	0.619	0.714	0.619	0.628	0.663	0.638	0.524	0.375
8	0.639	0.561	0.610	0.720	0.588	0.645	0.642	0.492	0.459
9	0.591	0.665	0.688	0.693	0.690	0.634	0.644	0.518	0.402
10	0.632	0.675	0.704	0.634	0.636	0.651	0.625	0.524	0.399
11	1.000	0.832	0.722	0.642	0.581	0.679	0.727	0.566	0.448
12	0.832	1.000	0.819	0.687	0.675	0.704	0.686	0.654	0.426
13	0.722	0.819	1.000	0.785	0.776	0.785	0.715	0.594	0.425
14	0.642	0.687	0.785	1.000	0.820	0.830	0.788	0.599	0.453
15	0.581	0.675	0.776	0.820	1.000	0.901	0.796	0.501	0.222
16	0.679	0.704	0.785	0.830	0.901	1.000	0.939	0.707	0.464
17	0.727	0.686	0.715	0.788	0.796	0.939	1.000	0.818	0.657
18	0.566	0.654	0.594	0.599	0.501	0.707	0.818	1.000	0.836
19	0.448	0.426	0.425	0.453	0.222	0.464	0.657	0.836	1.000

TABLE 3: Historically estimated forward rates correlation matrix

analogous surveys, see Rebonato (1998) or Jamshidian (1997). Many reasons can be put forward to explain such a difference: the use of correlations instead of covariances, the sample size, or, of course, an actually different market situation. In addition, the use of interpolation might have been relevant, even though our empirical tests do not underpin such hypothesis. However, the ultimate cause might be a simpler one. Indeed we consider 19 rates, whereas fewer variables are usually mentioned in defining the term structure. If we move to consider only 8 forward rates, 4 factors become enough to reach and exceed 91%. It appears that the number of initial forward rates included in the estimation is definitely relevant, whether it is due to added noise or to a real increase in dimensionality.

Now we have a realistic correlation benchmark, consistent with market tendencies, to be used as a blueprint for determining the values of the parameters in the presented correlation forms. The methods to do so are tackled in the following.

4.3 Pivot matrices

Here we concentrate on the parsimonious parameterizations seen in Section 4.1, that is on forms with two or three parameters. The classic methodology to find their parameters taking into account the estimation results, as proposed for instance by Schoenmakers and Coffey (2002), is fitting a parametric form to historical estimates by minimizing some loss function. But there is at least another procedure worth mentioning, resulting from the possibility to invert the functional structure of the parametric forms. In this way, we can express the parameters as functions of specific elements of the target matrix, so that such elements will be exactly reproduced in the resulting matrix. We can dub such elements "pivot points" of the target matrix, hence the resulting matrices will be referred to as pivot matrices.

If one is only looking for a parametric matrix as close as possible to the target one, the classic minimization procedure can be seen as the natural choice. Otherwise, there are several points that make the second proposal somewhat appealing:

1. It does not need any optimization routine, but can be carried out by simple, analytical formulas.

2. If the pivot points are chosen appropriately, it generates a behaviour similar to the one of the original matrix as regards the typical qualitative properties seen in Section 4.1.

3. It has a clear, intuitive meaning, since parameters are expressed in terms of correlation entries considered to be particularly significant. This allows us to easily alter and deform the matrix, as might be needed in the market practice for the reasons explained in Section 4, keeping under control the consequences.

4. It keeps out the negative effects of irregularities and clear outliers typical of historical estimations.

5. What is more, as will be seen, in our examples the extent of the overall error committed using this method is surprisingly close to the error in a complete, optimal fitting.

For these features to be all really present, the pivot points must be chosen carefully. We will start by considering three-parameters structures. Then, in our context, concerning forward rate correlations, particularly appealing are the entries $\rho_{1,2}$, $\rho_{1,M}$ and $\rho_{M-1,M}$, where M is the total number of forward rates involved. Such elements indeed embed basic information about the two typical properties of correlations amongst rates we have mentioned. In fact $\rho_{1,2}$ and $\rho_{1,M}$ are the first and the last of the values (different from one) on the first column/row, thus giving some good indication both on the extent and on the initial steepness of decorrelation along columns. In turn, $\rho_{1,2}$ and $\rho_{M-1,M}$ are the extremes of the first sub-diagonal, so acquainting us with some knowledge of the presence or not, and of the magnitude, of the increasing tendency along sub-diagonals.

We worked out the analytical relationships to build up the pivot versions of the three-parameters structures presented. Starting with Rebonato's form (13), we find the following expressions for the three parameters ρ_∞, α and β. First

$$\left(\frac{\rho_{1,M}-\rho_\infty}{1-\rho_\infty}\right) = \left(\frac{\rho_{M-1,M}-\rho_\infty}{1-\rho_\infty}\right)^{(M-1)},$$

from which one can, simply albeit numerically, extract the value for ρ_∞, entering the following relationships for α and β

$$\alpha = \frac{\ln\left(\dfrac{\rho_{1,2}-\rho_\infty}{\rho_{M-1,M}-\rho_\infty}\right)}{2-M}, \qquad \beta = \alpha - \ln\left(\frac{\rho_{1,2}-\rho_\infty}{1-\rho_\infty}\right).$$

Considering the numerical values for the three pivot correlations in the historical matrix, the results are

$$\rho_\infty = 0.23551, \quad \alpha = 0.00126, \quad \beta = 0.26388.$$

Let us now move on to form (10) by Schoenmakers and Coffey, called in short S&C3 in what follows. The first expression is as simple as

$$\rho_{M-1,M} = e^{-\beta},$$

while the remaining two, computationally slightly longer, finally read

$$\begin{aligned}
\alpha_1 &= \frac{6\ln\rho_{1,M}}{(M-1)\,(M-2)} - \frac{2\ln\rho_{M-1,M}}{(M-2)} - \frac{4\ln\rho_{1,2}}{(M-2)},\\
\alpha_2 &= -\frac{6\ln\rho_{1,M}}{(M-1)\,(M-2)} + \frac{4\ln\rho_{M-1,M}}{(M-2)} + \frac{2\ln\rho_{1,2}}{(M-2)},
\end{aligned}$$

leading to

$$\alpha_1 = 0.03923, \quad \alpha_2 = -0.03743, \quad \beta = 0.17897.$$

Also the pivot version of (11), called S&C2 in the following, will turn out to be useful. So, let us briefly expose the relationships obtained, using as pivot points $\rho_{1,M}$ and $\rho_{1,2}$. The first one is already a parameter of the form, being $\rho_\infty = \rho_{1,M}$. The second one is selected for reasons that will be clear later on in this section, when examining the behaviour of the matrices. With this choice, we get:

$$\eta = \frac{\left(-\ln \rho_{1,2}\right)(M-1) + \ln \rho_\infty}{2},$$

and consequently

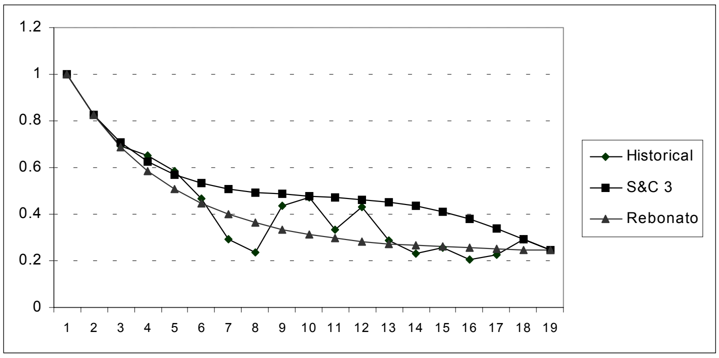
$$\rho_\infty = 0.24545, \quad \eta = 1.04617.$$

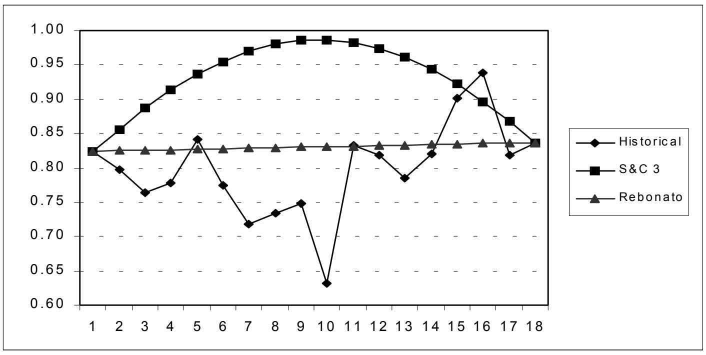
We compared the two three-parameters pivot forms with respect to the goodness of fit (to the historical matrix). We found that S&C3 pivot is superior when we take as loss function the simple average squared difference (denoted by MSE), whilst Rebonato pivot form is better if considering the average squared relative difference expressed in percentage terms with respect to the corresponding entry of the estimated matrix (denoted by MSE%). This is shown in the following table.

	MSE	MSE%	$\sqrt{\text{MSE}}$	$\sqrt{\text{MSE\%}}$
Reb. 3 pivot	0.030121	0.09542	0.173554	0.30890
S&C3 pivot	0.024127	0.10277	0.155327	0.32058

The latter measure (MSE%) appears to be particularly meaningful in our context, in that there are no entries so small as to lead to an overestimation of slight errors, while, due to the differences in magnitude of the correlation entries, the relative importance of the discrepancies can be more informative than their absolute value.

Some more reasons for considering Rebonato pivot form preferable in this context arise from the graphical observation of the behaviour of these matrices. As visible in the first figure below, showing the plot of the first columns, such matrix seems a better approximation of the estimated tendency, whereas S&C3 pivot tends to keep higher than the historical matrix. Moreover, in matching the estimated values selected, the parameter α_2 in S&C3 has turned out to be negative. This has led to a non-monotonic trend for sub-diagonals, see in fact the humped shape for the first sub-diagonal, plotted in the second one of the following figures.





A similar problem is hinted at also by Schoenmakers and Coffey (2000), who report that, in their constrained tests, α_2 tends to assume always the minimum value allowed, namely zero, and therefore they propose the form (11). We performed some more tests building different pivot versions of this last matrix. The results seem to suggest that this very faint increasing tendency along sub-diagonals, joined with the level of decorrelation along the columns seen in the historical estimate, represent a configuration very hard to replicate with this parameterization maintaining at the same time its typical qualitative properties. Indeed, building a pivot S&C2 keeping out information upon the sub-diagonal behaviour, one gets a matrix spontaneously featuring a strong increase along such sub-diagonals. On the other hand, including information on this estimated behaviour, a far larger decorrelation is implied than in the historically estimated matrix. More elements and details on such tests are given in Morini (2002). This simple example shows that the pivot methodology can prove useful to pick out some peculiarities of different parameterizations.

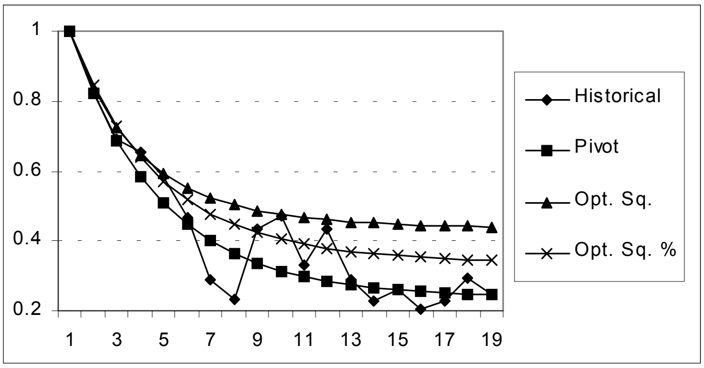
No such problem has emerged for Rebonato's form, that seems to allow for an easier separation of the tendency along sub-diagonal from the one along the columns. Moreover, notice that Rebonato pivot form, with our data, turns out to be positive definite, so that its main theoretical limitation does not represent a problem in practice.

Now we have still to check the divergence between pivot matrices and matrices optimally fitted to the entire target matrix. We have already seen that the error committed is reasonable, in particular considering that *only 3 points, out of 171, have been used.* Let us now see how it improves making use of the totality of information. We will compare the pivot version of Rebonato's parameterization with two optimal specifications of the same form obtained by minimizing the aforementioned loss functions. In the following table we present for each optimal form the square root of the corresponding error, besides the value obtained, for the same measure, when considering the pivot form.

	$\sqrt{\text{MSE}}$	$\sqrt{\text{MSE}\%}$
Fitted vs Historical	0.108434	0.25949
Pivot vs Historical	0.173554	0.30890

Apparently, *the addition of 168 data to the 3 pivot points* brought about quite a narrow improvement, especially as regards the percentage error, particularly meaningful in our context. This is confirmed by the graphical comparison of the columns, as visible, for instance, in the figure below, plotting first columns.

Overall, considering also the other columns, no parameterization stands clearly out.



The improvement seen in the error cannot be negligible, if one is only looking for a replication of the target matrix as accurate as possible. On the contrary, this is not really noteworthy when what is needed is mainly a fast and intuitive way to build up a correlation matrix, if possible synthetically consistent with major market tendencies, but above all regular and easy to modify. This is just the case in the following cascade calibration market tests.

5 Cases of calibration

5.1 Various correlations and ranks

The swaption market data used for these calibration cases are those in Table 2, thus completed by local linear interpolation and referring to February 1, 2002. At first we will consider the results of calibration to only the upper part of the swaption matrix, since results for the remaining part are strongly influenced by the hypothesis made for reducing the number of unknowns.

The first correlation matrix we apply, in view of the results of the previous section, is Rebonato 3 parameters pivot. We will consider different ranks, obtained by fitting the angles form seen in Section 2.2, at the desired rank, onto the full rank parsimonious parameterization. For instance, we can recall the principal component analysis we carried out and decide to start with rank 7. The calibrated volatilities are

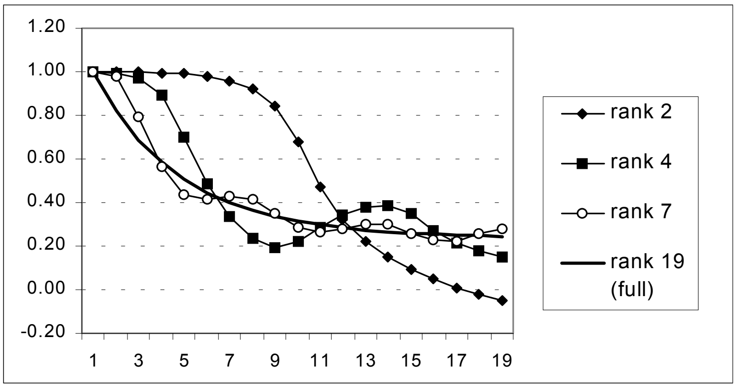
0.179									
0.153	0.155								
0.144	0.129	0.154							
0.144	0.134	0.105	0.156						
0.140	0.122	0.112	0.112	0.154					
0.143	0.134	0.103	0.101	0.106	0.153				
0.143	0.127	0.143	0.088	0.097	0.086	0.144			
0.146	0.153	0.128	0.078	0.070	0.098	0.093	0.145		
0.157	0.109	0.155	0.160	0.067	0.007	0.101	0.081	0.107	
0.136	0.152	0.126	0.123	0.121	0.108	-0.040	0.120	0.077	0.067

So there is a negative volatility, $\sigma_{10,7}$. What can we do to avoid this problem? We can act on the swaption volatilities, or else on the correlations. Here we consider the latter opportunity, therefore let us start by changing the rank of the correlation matrix. Considering that similar problems were present in the two-rank tests by Brigo and Mercurio (2001), we might suspect reduced rank to be responsible. But a calibration with full rank, equal to 19, gives us not only the same negative volatility, but also a complex one, $\sigma_{10,10}$.

Let us then try and reduce the rank. Down to rank 5 we get the same negative volatility, though reduced in absolute value. At rank 4 it disappears, and the output is completely acceptable, as visible in the following table.

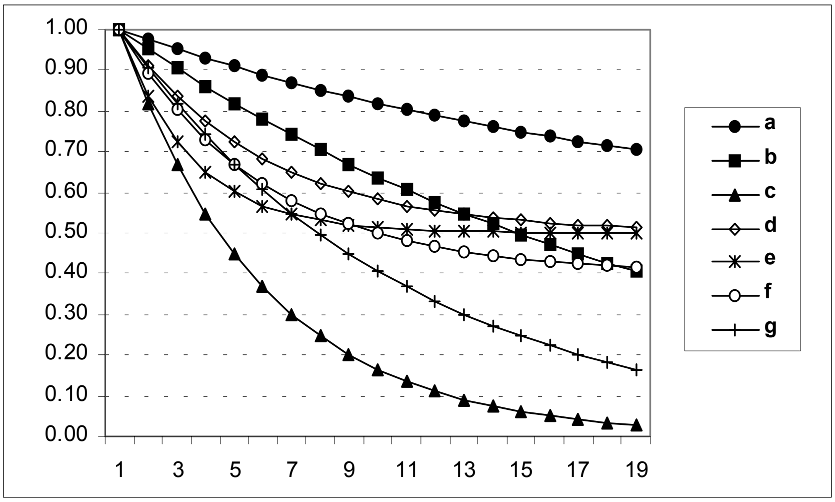
0.179									
0.152	0.156								
0.131	0.130	0.165							
0.123	0.132	0.120	0.164						
0.128	0.123	0.120	0.118	0.153					
0.141	0.128	0.098	0.101	0.108	0.162				
0.144	0.115	0.122	0.082	0.102	0.106	0.159			
0.147	0.137	0.106	0.065	0.071	0.110	0.114	0.159		
0.156	0.098	0.136	0.131	0.054	0.031	0.119	0.111	0.139	
0.134	0.147	0.117	0.106	0.095	0.086	0.007	0.138	0.102	0.122

The same happens for rank 3 and 2. What might cause a similar behaviour? Recall that lowering the rank of a correlation matrix amounts to impose an oscillating tendency to the columns, that for very low ranks is represented by a sigmoid-like shape. This is clear in the figure below, showing the first column of Rebonato 3 parameters pivot correlation matrix at different ranks.



A strong interpretation of such results might be based on the fact that 4 is also the number of relevant factors we find in the PCA shrunk to a classic number of initial forward rates. A less far-reaching, but more apparent interpretation simply suggests that some features of the lower rank correlations are better suited to these swaptions data. In particular, the most relevant feature discriminating between lower and higher rank matrices seems to be the slope of the columns in their initial part, that is the initial steepness of the decorrelation among rates. We might then elicit that correlation matrices characterized by less steep initial decorrelation allow for acceptable results, whereas those featuring more marked steepness force some σ's to assume weird values.

But a similar interpretation must be underpinned by some more evidence before we can consider it a sound one. Therefore we made a number of further tests, especially making use of simple, synthetic correlation matrices, whose essential features can be easily modified and controlled. Let us then see how the calibrated volatilities change when varying the parameters of the classic exponential structure given in (12). We varied the two parameters α and β so as to obtain different configurations in terms of correlation patterns, represented in the figure below, where we report, as usual, the first columns. A detailed description of the parameters values giving rise to these configurations and a longer discussion on these tests can be found in Morini (2002).



We start with the matrix whose first column is represented by **a**, obtained by setting $\alpha = 0.5$ and $\beta = 0.05$, as in Brigo (2002). With such a correlation, at full rank we obtain volatilities all real and positive, even calibrating to the entire swaption matrix, though in the following we will go back to considering only the upper part. Then we lower the rank, first to 15 and successively to 5, a level we keep in the following because representing the first problematic level when increasing the rank of Rebonato three-parameters form. Also with such ranks, we find only perfectly acceptable results.

Then we move α and β, producing all the configurations shown, different in terms of extent of the decorrelation, initial steepness, and final level reached by correlation. With the correlations corresponding to **b**, **d** and **g**, we keep avoiding negative or complex volatilities, whereas **c**, **e** and **f** give a negative volatility in the same position as in our previous tests, that is $\sigma_{10,7}$. It is apparent that neither the final level of correlation nor the inclination of the terminal part of the columns are decisive features to distinguish problematic from non-problematic correlations. Instead, what shows up is that we find non-meaningful results for those correlations featuring columns initially steeper, while the four configurations characterized by less initial steepness result in real and positive volatilities. This seems to underpin our previous hypothesis that what gives problems with these swaption volatilities is initial steepness. In other words, we might say that, in this context, correlations of such a kind are not consistent with the swaption market data, completed by local linear interpolation, used for our tests. We will go back to this possible use of cascade calibration as a tool to detect inconsistency between model structures and market data in Section 5.2 and in the conclusions.

Now, let us see if some previously presented parametric form can allow for avoiding problems appeared using Rebonato 3 parameters. Based on the results above, S&C2 pivot looks interesting, since, as we alluded to, is characterized by a more pronounced increase along sub-diagonals, feature here coupled with, in general, initially less steep decorrelation along columns. We remark that for columns different from the first one, "initial" means starting by the unitary element on the principal diagonal. As one might by now expect, actually this correlation gives us volatilities all real and positive, even at full 19 rank. And of course this property is preserved also when reducing the rank. In particular, for rank 2, we do not have nonsensical correlations even if we calibrate to the entire matrix, as shown in the table below.

0.179									
0.152	0.156								
0.130	0.130	0.166							
0.119	0.131	0.122	0.167						
0.112	0.115	0.120	0.126	0.164					
0.112	0.115	0.100	0.113	0.126	0.171				
0.113	0.103	0.119	0.098	0.120	0.119	0.163			
0.122	0.124	0.108	0.082	0.091	0.121	0.119	0.160		
0.138	0.093	0.130	0.129	0.073	0.047	0.123	0.113	0.149	
0.121	0.129	0.106	0.098	0.092	0.090	0.023	0.144	0.118	0.147
0.120	0.120	0.101	0.093	0.134	0.063	0.060	0.045	0.142	0.108
0.107	0.107	0.107	0.142	0.036	0.135	0.078	0.063	0.051	0.143
0.112	0.112	0.112	0.112	0.084	0.084	0.074	0.108	0.062	0.052
0.103	0.103	0.103	0.103	0.103	0.123	0.116	0.043	0.105	0.061
0.097	0.097	0.097	0.097	0.097	0.097	0.169	0.088	0.068	0.108
0.093	0.093	0.093	0.093	0.093	0.093	0.093	0.153	0.117	0.089
0.094	0.094	0.094	0.094	0.094	0.094	0.094	0.094	0.090	0.155
0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.016
0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099

Although finding out that it is possible to detect typical correlation features avoiding the common problems of cascade algorithms is a comforting result, it is worthwhile reminding that such results depend on the particular market quotations we had available, and similar analysis should be carried out again for markedly different market situations. Moreover, we remark that intermediate configurations, with respect to the features we considered to be decisive, might give rise to less clear results, possibly due to the influence of some different, less evident factor. Finally, these findings depend also on the interpolation used for missing market quotations. This is the issue addressed in the next section.

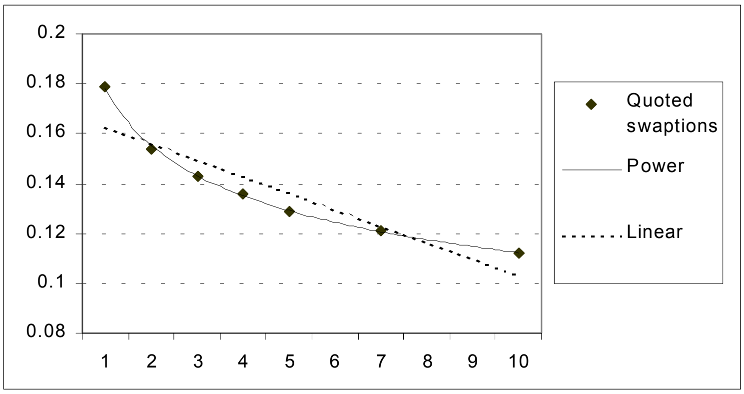
5.2 The interpolation for the swaption matrix

We can start to consider this point with an interesting remark: in all previous cascade calibration tests, including those in Brigo and Mercurio (2001) and (2002), *negative or complex values occur only for volatilities depending also on the artificial swaption volatilities obtained by a local linear interpolation along the columns of the swaption matrix. On the contrary, volatilities obtained before such artificial values enter the algorithm are all real and positive.*

Consequently, we are naturally led to suspect the interpolation used for missing market quotations to bear some responsibility for the problems found. The first step might be checking whether the linear interpolation is really the most suited to replicate the typical patterns in the swaption market. We tried out some fitting forms for the columns of the swaption matrix. Considering the trade-off between goodness of fit and simplicity, we will show the results for a power interpolation whose functional form, for the first column, is

$$Y = 0.1785\,(X)^{-0.201}\,,$$

where Y denotes the swaption volatility and X the maturity. Remark that, since we started by considering the issue of the interpolation employed for missing data, we at times use the term "interpolation" also when referring to the fitting form we use first for the same purpose and later for replacing particular market values. In the figure below, market quotations, the power (or log-linear) fitting form and the linear one are shown. Notice that both linear and power fittings considered are "global", that is non-local, since based on all available values.



The power fitting form appears clearly closer to the real market pattern than the linear one. This is confirmed also by checking the values for the common R^2 index: $R^2 = 0.999$ for the power form, compared to $R^2 = 0.836$ for the linear one. Also a graphical comparison regarding the other columns confirms the superiority of the power form. In order to make sure this was not a one-off coincidence, we tried the same with quotations referring to some months later, finding analogous results.

However, we must recall what is reported in Rebonato and Joshi (2001) about typical swaption configurations. According to this work, two are the common shape patterns that can be found in the Euro swaption market: a humped one, called *normal* and typical of periods of stability, and a monotonically decreasing one, called *excited* since associated with periods immediately following large movement in the yield curve and in the swaption matrix. Our data appear easily to belong to the second pattern. Of course, in periods characterized by humped patterns, a similar form would be likely to prove inadequate.

It is natural to wonder whether, using such a more realistic interpolation for missing maturities, it is possible to change the output of the cascade calibration. We perform a calibration interpolating with the power form above, and keeping as correlation the Rebonato three-parameters pivot form. The negative entry we found when using linear local interpolation does not disappear, but reduces in absolute value, confirming that a more realistic interpolation can affect the output, and for the better. More refined interpolations, and in particular local interpolations, i.e. based only on a few quotations adjacent to the missing maturities, might produce further improvements. A complete test of all possibilities would be beyond the scope of this work, but we suggest this issue of an interpolation really consistent with market quotations to be an interesting field for further research. In such a case, of course, "consistency" is evaluated observing the output of the cascade calibration, representing, via the common industry approximation, the pieces of forward rates volatilities implied by the calibration inputs.

But the global power interpolation above has turned out to be particularly useful for a broader application. In fact, the position of the negative value in the above-mentioned tests makes it particularly dependent on swaption volatilities with maturity of 7 years. This maturity is peculiar, because isolated between missing quotations. And experienced market operator confirm it to be rather illiquid (Dominici (2002)), and hence prone to possible misalignments. Brigo and Mercurio (2001) and (2002) already suggest that negative or complex volatilities might be due to illiquidity and temporal misalignments in the swaption matrix, and propose a general smoothing to get rid of such values.

Here, taking advantage of previous calibration results, market information and the global alternative interpolation introduced, we propose a different, more specific solution. Only the quotations for swaptions with maturity of 7 years are replaced, and the functional form used for such values and for the missing ones is the power form seen above. In this way, the influence on the original matrix is far and away less drastic, in that, when considering a calibration to the upper part of the swaption matrix, only 4 (instead of 55) values have been substituted. Further, the errors induced for the replaced values are definitely lower, as we can see in the table below:

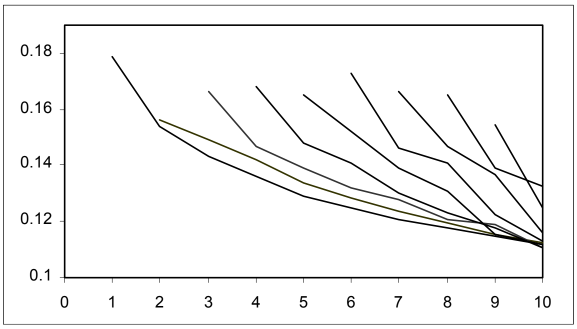
Errors	-0.00028	-0.00119	-0.00079	0.00049
% Errors	-0.23272%	-1.03388%	-0.70780%	0.45776%

If misalignments on the seventh row are really causing problems, and the use of a more realistic fitting form for missing and misaligned values can really be of help, then even this limited substitution should bring about some improvement in calibration. And this is indeed the case: keeping our first choice for the correlation matrix, Rebonato three-parameters pivot, and calibrating to the upper part of the swaption matrix, the previously found negative value disappears at any rank. Even reaching full 19 rank, all volatilities are real and positive. It seems that, in our examples, with a correlation matrix reflecting some typical features of the historical estimation, namely a steep initial decorrelation along columns, calibration is particularly sensitive to irregularities and inconsistencies in the input swaption matrix. Hence an accurate use of interpolation techniques for missing or misaligned data can improve the output. Once more we see a possible use of this analytical calibration for detecting inconsistencies, in this case among market and artificial values, given a particular correlation matrix.

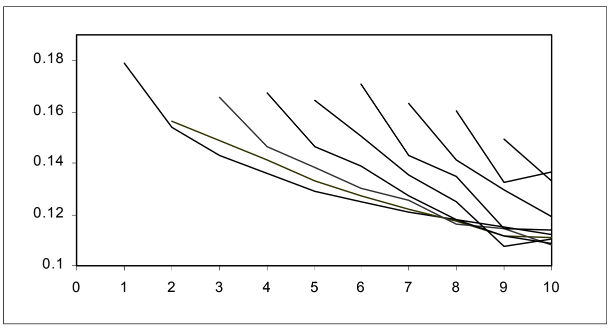
5.3 Checking the TC and the evolution of the TSV

The LMM is extremely rich of volatility and correlation parameters, and this tempts the user to treat such quantities as pure free-fitting parameters, irrespective of any consideration on their configuration, internal consistency, and structural implications. In order to keep away from such a danger, Rebonato (1998) suggests to impose structural constraints on the shape of covariance structures and to make use of all possible checks of consistency of different quantities. In line with similar remarks, Brigo and Mercurio (2001) focus on the need for a calibration allowing for instantaneous and terminal correlations, and the evolution of the term structure of volatilities, being meaningful and regular, in the sense of the requirements mentioned in Section 5.3.

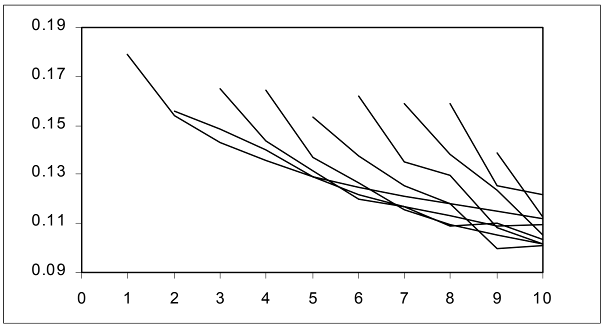
In accordance, here we examine the configurations assumed by the above structures in some of the market calibrations we performed. Let us start by considering the evolution of the TSV. First, we see below how it appears in case of a calibration with Rebonato three-parameters pivot correlation matrix at rank 2.

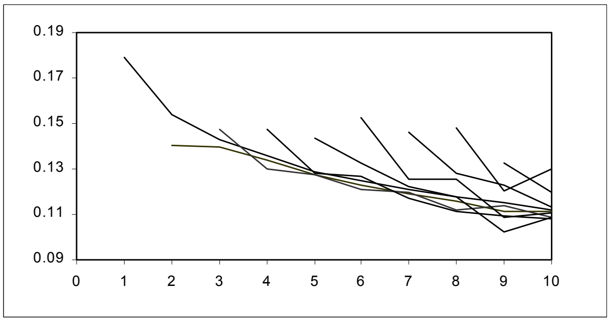


Considering the generality of the GPC volatility parameterization, such evolution appears surprisingly regular, smooth and stable over time, as well as being rather realistic. In fact, although it features no hump, such a characteristic is consistent with the swaptions data we are calibrating to. These are the properties we are usually looking for. In order to assess if the form chosen for the correlation has been really relevant, we show in the figure below the evolution we obtain with a S&C2, keeping the same rank.



Although the general features we appreciated earlier are still present, this less flexible form has brought about some worsening. Let us see what happens when increasing the rank. We plot now the results with Rebonato pivot at rank 4, and immediately below with S&C2 pivot at rank 10.





It is apparent, in both cases, that the more realistic and detailed correlations we get with higher rank can cause less regular and less stable evolutions of the term structure of volatilities. This is confirmed also by other tests we performed, with similar correlations and even higher rank. However, we underline that, as one might expect, using particularly smooth and stylized correlations it is possible to attain a regular evolution even at full rank.

Now we move on to examining terminal correlations. Following Brigo and Mercurio (2001), we present the TC after ten years. In the following tests, we stick to S&C2 pivot instantaneous correlation matrix, trying out different ranks. Starting with rank 2, we obtain the following matrix:

	10	11	12	13	14	15	16	17
10	1.000	0.928	0.895	0.920	0.855	0.846	0.928	0.924
11	0.928	1.000	0.863	0.909	0.933	0.881	0.901	0.923
12	0.895	0.863	1.000	0.916	0.908	0.910	0.878	0.939
13	0.920	0.909	0.916	1.000	0.944	0.931	0.956	0.926
14	0.855	0.933	0.908	0.944	1.000	0.954	0.923	0.928
15	0.846	0.881	0.910	0.931	0.954	1.000	0.937	0.958
16	0.928	0.901	0.878	0.956	0.923	0.937	1.000	0.957
17	0.924	0.923	0.939	0.926	0.928	0.958	0.957	1.000

Such a matrix is definitely far from being ideal. However, we must remark that the peculiarities of S&C2 matrix and the severe rank reduction result in an instantaneous correlation matrix with values in the corresponding portion already very high and close to each other. The terminal one is only replicating such features perturbed by the instantaneous volatilities dynamics. Thus to get more meaningful results we must increase the rank, for instance up to ten, as it is for the ten-year TC matrix below.

	10	11	12	13	14	15	16	17
10	1.000	0.887	0.806	0.792	0.708	0.690	0.757	0.734
11	0.887	1.000	0.822	0.837	0.825	0.746	0.749	0.753
12	0.806	0.822	1.000	0.877	0.841	0.820	0.758	0.801
13	0.792	0.837	0.877	1.000	0.919	0.877	0.881	0.806
14	0.708	0.825	0.841	0.919	1.000	0.932	0.878	0.840
15	0.690	0.746	0.820	0.877	0.932	1.000	0.915	0.914
16	0.757	0.749	0.758	0.881	0.878	0.915	1.000	0.934
17	0.734	0.753	0.801	0.806	0.840	0.914	0.934	1.000

This matrix looks in general satisfactory, and the same applies to the matrix we get when going up to full rank. The typical forward rates correlation patterns described earlier are clearly visible, following our qualitative desiderata. Moreover, such patterns look still very similar to those in the instantaneous correlation matrix, although time-varying volatilities, while reducing correlations, have moderately perturbed the typical properties. Similar results have been obtained also when using different forms for instantaneous correlations.

These checks seem to confirm that the automatic calibration methodology used here can generate parameters not only assuring a perfect fit, but also regular and meaningful. Moreover, such results suggest that there might be a trade-off between regularity of the evolution of the TSV and satisfactory TC's, in that these features vary in opposite directions when rank is changed. Obviously, for different data and different parameterizations one should check again if such a trade-off is still relevant. In such a case, a possible criterion to make a choice is considering whether the financial instruments one is applying the model to depend essentially on instantaneous volatilities or are strongly influenced by the configuration of correlations.

6 Monte Carlo tests

We have already pointed out that the reliability of the exact swaptions cascade calibration depends also on the accuracy of the underlying approximated formula (5). This formula has already been tested, for instance by Brigo and Mercurio (2001) and Jäckel and Rebonato (2000). What we do here is extending similar tests, based on Monte Carlo simulation of the true LMM dynamics, to different or more general conditions, either peculiar to the cascade calibration, such as GPC volatility structure, or particularly easy to implement in this context, such as various correlations with various ranks.

First of all, we need to discretize the dynamics seen in Section 2.1, in particular (1). Taking logs in order to get the stronger convergence of Milstein scheme, one obtains

$$\begin{aligned}\ln \bar{F}_k(t+\Delta t) = \ln \bar{F}_k(t) + \sigma_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)\bar{F}_j(t)}{1+\tau_j\bar{F}_j(t)}\Delta t \\ - \frac{\sigma_k(t)^2}{2}\Delta t + \sigma_k(t)(Z_k(t+\Delta t) - Z_k(t)).\end{aligned} \tag{14}$$

Around the Monte Carlo price we get, a two-side 98% window is built, based on the standard error of the method. Then, by inverting Black's formula, we obtain the corresponding values for volatilities, which are compared with the result of formula (5). In particular, we check if the approximation is in-between the extremes of the Monte Carlo window, denoted by "inf" and "sup". In most cases, input volatilities are the output of a cascade calibration, so that formula (5) yields exactly the market original value. In such cases, this is what is compared with the value implied by the true LMM Monte Carlo dynamics specified by the same covariance parameters.

All following tests refer to simulations with 4 time-steps per year and 200000 paths, "doubled" by the use of the variance-reduction technique known as antithetic variates. We selected a sample of test results, particularly meaningful since covering a differentiated range of conditions.

a) Let us start by checking how the approximation works under GPC parameterization with low rank and "normal" conditions as for swaption characteristics, so that we consider a 5×6 swaption with Rebonato pivot matrix at rank 2. We obtain:

MC volatility	MC inf	MC sup	Approximation
0.108612	0.108112	0.109112	0.109000

b) Now we try out an increase in swaption maturity and underlying swap length. Indeed we consider a 10×10 swaption, with S&C2 matrix at rank 2, obtaining:

MC volatility	MC inf	MC sup	Approximation
0.097970	0.097531	0.098409	0.098000

c) We see here a little higher rank, with a 5×6 swaption and S&C 2 correlation matrix now at rank 5.

MC volatility	MC inf	MC sup	Approximation
0.108824	0.108323	0.109325	0.109000

In the three cases we considered, with market calibrated volatilities, the volatility approximation (5) appears definitely good. This confirms its accuracy also in case of GPC volatility, with low or intermediate rank, and even for long maturities and lengths.

d) We move now on to consider artificially modified instantaneous volatilities, so as to test the approximation in case of anomalous values. First, we upwardly shift all calibrated volatilities ("Cal. vol's" in the following tables) by multiplying them by 1.2. Then we experiment stronger increase, by adding 0.2 to all volatilities, following Brigo and Mercurio's (2001) testing plans. In this case, it appears useful to show also what is returned by a simple Monte Carlo without variance-reduction techniques, denoted by "MC vol (no a.v.)". The results, maintaining the remaining conditions as in test a), are as follows:

	MC volatility	MC inf	MC sup	Approx.
Cal. vol's $\times 1.2$	0.130261	0.129644	0.130878	0.130800
Cal. vol's $+0.2$	0.300693	0.298988	0.302399	0.303820
	MC vol (no a.v.)	MC inf	MC sup	Approx.
Cal. vol's $+0.2$	0.301934	0.299010	0.304861	0.303820

e) Let us perform the same set of tests, but this time on a 6×7 swaption, with S&C 2 correlation matrix and rank increased up to 10. We obtain:

	MC volatility	MC inf	MC sup	Approx.
Cal. vol's	0.104608	0.104131	0.105085	0.105000
Cal. vol's $\times 1.2$	0.125469	0.124881	0.126057	0.126000
Cal. vol's $+0.2$	0.286702	0.285133	0.288272	0.289777
	MC vol (no a.v.)	MC inf	MC sup	Approx.
Cal. vol's $+0.2$	0.287380	0.284671	0.290091	0.289777

Both in point d) and e) we see that even with moderately increased volatilities, the approximated formula still works well in this context of GPC volatilities, different correlations and low or intermediate rank. In case of very high volatilities, the approximation is outside the antithetic variates Monte Carlo window, but inside the non-reduced variance simple Monte Carlo. Therefore, we conclude that the approximation keeps even in this case an acceptable behaviour, but looses accuracy.

f) Then we performed some tests in more extreme conditions. See below the results obtained with a 10×10 swaption, classic exponential correlation (12) as in Brigo (2002), and full 19 rank.

	MC volatility	MC inf	MC sup	Approx.
Cal. vol's	0.097764	0.097327	0.098202	0.098000
Cal. vol's $\times 1.2$	0.117215	0.116681	0.117750	0.117600
Cal. vol's $+0.2$	0.285596	0.284190	0.287003	0.290606
	MC vol (no a.v.)	MC inf	MC sup	Approx.
Cal. vol's $+0.2$	0.285941	0.283391	0.288497	0.290606

In this case of maximum rank coupled with a "far" and "long" swaption, the approximation appears good for normal or slightly increased volatilities, but gets less reliable for very high volatilities, being outside both MC windows. Further tests of ours seem to suggest that the factors that most influenced such a worsening are the elevated maturity and length, rather than the high rank.

g) In this last set of tests, we investigate the case of higher input forward rates. Thus we uniformly upwardly shifted the initial forward-rates vector by adding 0.02 to all original rates. Initially we apply that to the same covariance structure as in test c), obtaining:

MC volatility	MC inf	MC sup	Approximation
0.108923	0.108437	0.109410	0.109136

h) And then we consider the extreme situation of point f), with the following results:

MC volatility	MC inf	MC sup	Approximation
0.097716	0.097301	0.098130	0.098048

In the typical context of cascade calibration, the approximation (5) appears reliable even with higher forward rates, irrespective of the level of the rank and the features of the swaption considered.

To sum up, our tests seem to allow us to extend the generally positive judgement on the reliability of industry approximation (5) also to configurations typical of this work such as GPC parameterization and different correlation structures with various ranks, ranging from 2 to 19. This holds true even for high maturity and length swaptions, moderately increased volatilities and upwardly shifted initial forward rates. On the other hand, we can also extend to such cases the more negative valuation expressed for instance by Brigo and Mercurio (2001) on the accuracy of the formula in case of market situations characterized by very high volatility that, following again Brigo and Mercurio (2001), we might consider pathological.

Similar remarks apply therefore also to the automatic swaptions cascade calibration, which is based on this formula joined with GPC volatilities and exogenous correlations.

7 Considering the cap market

Even though some sound objections have recently been raised to the convenience of carrying out joint calibrations to cap and swaptions, as we will see in the following, one might find it interesting to consider how information coming from cap data can be embedded in this calibration, or at least how the two sets of information can be given consistency. From our experience, this a typical question traders ask when this methodology is presented.

In the following, we provide some hints and discussions on the matter. The first point to address is the annualization of semi-annual caps data, so as to make them consistent with usually annual swaptions data. Consider three instants $0 < S < T < U$, all six-months spaced, and assume we are dealing with an $S \times 1$ swaption and with S and T-expiry six-month caplets. Let us denote by v^2_{Black} the Black's swaption volatility and by $\sigma_1(t)$ and $\sigma_2(t)$, respectively, the instantaneous volatilities of the two semi-annual forward rates $F_1(t)$ and $F_2(t)$ associated with the two caplets, whereas $F(t)$ is the annual S-expiry forward rate. Then we will make use of the following relationship to connect the above quantities:

$$\begin{aligned} v^2_{\text{Black}} \approx \tfrac{1}{S} \Big[& u_1^2(0) \textstyle\int_0^S \sigma_1(t)^2 \, dt + u_2^2(0) \int_0^S \sigma_2(t)^2 \, dt \\ & + 2\rho u_1(0) u_2(0) \textstyle\int_0^S \sigma_1(t)\sigma_2(t) \, dt \Big], \end{aligned} \tag{15}$$

where

$$\begin{aligned} u_1(t) &= \frac{1}{F(t)} \left(\frac{F_1(t)}{2} + \frac{F_1(t)F_2(t)}{4} \right), \\ u_2(t) &= \frac{1}{F(t)} \left(\frac{F_2(t)}{2} + \frac{F_1(t)F_2(t)}{4} \right), \end{aligned}$$

and ρ represents the *infra-correlation* between the two semi-annual forward rates. For a derivation, based mainly on no-arbitrage relations between forward rates and on the "freezing" approximation of the u's, see Brigo and Mercurio (2001). In that work is also shown that, assuming forward rates have constant volatilities, formula (15) reduces to

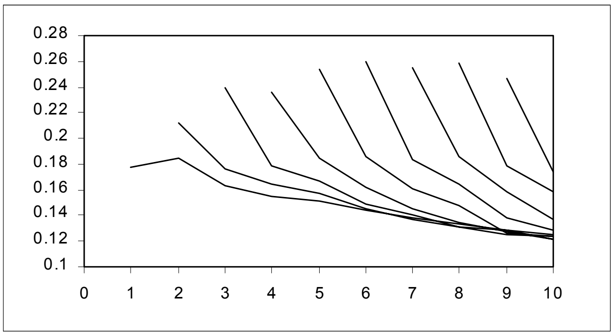
$$v^2_{\text{Black}} \approx u_1^2(0) v^2_{S-caplet} + u_2^2(0) v^2_{T-caplet} + 2\rho u_1(0) u_2(0) v_{S-caplet} v_{T-caplet}, \tag{16}$$

where $v_{T-caplet}$ denotes the volatility of the T-expiry caplet.

Now we can introduce a simple, rather brute-force opportunity to incorporate cap data into the cascade calibration. In fact, given the formulas above, we can simply replace the first column of the input swaption matrix, containing volatilities for unitary length swaptions, with the corresponding array of annualized caplet volatilities. In this way, we actually lose some information on the swaption market, but in turn we achieve a perfect fit to annualized caplet volatilities. To implement such an approach with our data, referring as usual to February 1, 2002, we made use of formula (16), in that it depends only on market quantities, taking all ρ's equal to one, as in Brigo and Mercurio (2001). Some more elements on the tests of this section can be found in Morini (2002). We obtain

Annualized caplet volatilities									
0.178	0.185	0.163	0.155	0.152	0.145	0.138	0.134	0.129	0.125

Except the first one, these values are all higher than the swaption volatilities they are to replace. They have been included in a calibration with Rebonato three-parameters pivot correlation matrix at rank 4, the same correlation we used in a test of Section 5.1. The calibrated σ's we obtain now are still real and positive, but many entries are markedly different from the corresponding ones in the previous test mentioned. What is more, the implied evolution of the term structure of volatility turns out to be smooth but definitely peculiar, as can be seen in the following figure.



Although we do not get nonsensical instantaneous volatilities, a similar evolution of the TSV can cause some perplexity, leading to suspect it might be due to a possible inconsistency between the caps and swaptions data we mixed together. However, we should consider that a similar result is found also in Brigo, Mercurio and Morini (2002), where the original all-swaption matrix simply featured a humped shape in the first column, like our matrix does after the substitution. Moreover, Brigo and Mercurio (2001) point out that formula (16) tends to overprice swaptions, a bias that might have been further increased by fixing infra-correlations to their maximum value. Consequently, better-founded conclusions might be induced by further investigations, including different, more realistic assumptions on instantaneous volatilities, and different values for the ρ's, for instance based on the analysis of historical or implied typical infra-correlation patterns.

The latter point is partially addressed in the following. In fact, we use the above setting to calculate the infra-correlations implied in our data, since this can be seen as a different opportunity to link a swaptions cascade calibration with the information coming from the cap market. Indeed, the calibration considered earlier in this work provides us with an annual Libor Market Model exactly and automatically calibrated to swaptions. Now this can be coupled with a semi-annual LMM exactly and automatically calibrated to caps, being the relationships between the two models represented by a specification of formula (15), and hence by the corresponding values for the infra-correlation parameters ρ. For example, if the specification chosen is formula (16), then extracting v^2_{Black} from the swaption market and $v_{S-caplet}$ and $v_{T-caplet}$ from the cap market we can invert the formula easily obtaining the value for ρ. This is the scheme we implemented, obtaining the following ρ's, each one associated with a different unitary swaption:

Infra-correlations									
1.022	0.388	0.543	0.536	0.444	0.493	0.533	0.560	0.586	0.598

If one deeply trusts the model and the formulas presented above, such results might be considered a possible confirmation of the underlying assumptions, in particular the existence of a well-defined relationship between cap and swaption volatilities. In fact, the values we obtained are indeed viable correlations, except for the first one. And also the anomaly regarding that value is not particularly relevant. From such a viewpoint, this might even be considered a possible arbitrage opportunity to exploit.

However, a more cautious and critical approach is likely to give rise to a different kind of remarks. First, besides the fact that the first value is outside the viable range for correlations, the other values appear too low to represent real correlations between adjacent rates. Although the variables are different, just compare them to the results of our historical estimation. A possible reason for this is the aforementioned bias due to the chosen volatility parameterization. Again, more realistic hypothesis can lead to different results. But the really relevant reasons calling for a cautious interpretation of such results are of a different nature. Indeed, relations and discrepancies between caps and swaptions tend to be influenced by causes concerning the market fundamentals. This point is closely related to a more general problem, namely whether or not there exists a basic congruence between the cap and swaption markets, that a model can successfully detect and incorporate. Rebonato (2002) seems to warn against excessive enthusiasm in considering such a possibility. He recalls that problems such as illiquidities, agency problems and value-at-risk based limit structures strongly reduce the effectiveness of those market operators, called *quasi-arbitrageurs*, that are supposed to maintain the internal consistency between the two markets. Thus swaption and caplet volatility surfaces may turn out to be non-congruent.

Accordingly, simple artificial values such as the infra-correlations above, expressing synthetically the relationships between those surfaces in the context of a modeling framework, are likely to be actually influenced by many different external factors that are hard to detect and measure. Maybe some more light might be shed on this matter by comparing implied and historically estimated infra-correlations, and analyzing their stability over time.

To conclude, even more caution is required in considering whether to apply similar outputs to a trading strategy. We recall the words by Rebonato, quoted in Alexander (2002), suggesting that apparent arbitrages arising from mispricing of either caps or swaptions by a general model would be too risky to trade upon.

And a further warning comes from recalling that the losses involving several banks at the end of the 90's can be attributed to the fact that the used models relied on a fixed relation between caplet and swaption volatilities. As recalled by Pelsser (2000), such an assumption was finally belied by the true market behaviour.

8 Conclusions

In our discussions, different issues regarding the Libor Market Model automatic swaptions cascade calibration have been considered. After recalling the basic facts, we introduced explicitly a possible algorithm to extend the calibration to the entire swaption matrix. We remarked that some fundamental features make this methodology particularly appealing: it is automatic and analytical, and hence instantaneous; if a common industry approximation is used for pricing, it is free from any calibration error; it allows for a direct correspondence between market swaption volatilities and LMM volatility parameters. We pointed out that a further opportunity is given by the exogenous nature of the forward rates correlation matrix. Accordingly, we both estimated historical correlation and considered regular and parsimonious parameterizations, being led to a simple and intuitive methodology to fix parameters consistently with general market tendencies. In this way instantaneous correlation matrices that are rather realistic, regular and simple to control and modify can be easily obtained. Moreover, as we showed, regular terminal correlations and a satisfactory evolution of the term structure of volatilities are possible, even though our tests revealed a possible trade-off between regularity of the evolution of the TSV and realism of TC's, depending on the level of the rank.

The two questions that might prevent from claiming such a calibration to be completely satisfactory for practical purposes have both been dealt with in this work. The first one is the reliability of the industry approximation this method is based on, under the covariance structures typical of this context. We tested it via Monte Carlo simulation, concluding the approximation to be sufficiently accurate even under these different conditions, except in case of anomalously high volatility conditions. Thus such a generally positive conclusion can be extended also to the calibration method considered.

The second problem is the possible occurrence of nonsensical results, such as negative or complex volatility parameters, as typical in previous literature. Here we identify a number of correlation structures giving rise to perfectly acceptable results, pointing out empirical links between some major characteristics of the correlation matrix, namely a steep initial decrease along columns, and the occurrence of anomalous outputs. Then we address the related matter of the interpolation used for missing market quotation, and propose an alternative power form appearing much closer to real market patterns than the usual linear interpolation. Such power fitting form turns out to be especially useful for replacing values deemed to be misaligned, and as such responsible for anomalous results. In fact, this substitution leads to positive real volatilities, meanwhile allowing for a calibration that is far more accurate than the general smoothing calibration previously proposed to overcome this problem.

Also these results are comforting, in that they prove that it is possible, given a set of market data, to identify typical configurations leading to acceptable parameters. However, this might not yet be considered enough for an automatic, trouble-free, daily market application. On the other hand, the possible occurrence of anomalous results can be seen as an alarm bell helping us to detect inconsistencies in the calibration. From such a viewpoint, these algorithms, providing us with pieces of forward rates volatilities kind of implied by swaption quotations, given a correlation structure and via the filter represented by the common industry approximation, can be considered a fast tool for recognizing consistency problems between the different inputs of the calibration. This is the kind of use we often made of cascade calibration in the previous tests.

Finally, we took some steps into the opportunities to consider cap quotations in this context, consequently hinting at the problem of finding, if existing, meaningful relationships between swaption and cap markets. This can be an interesting field for further research, for instance starting with a detailed analysis, possibly involving historical estimation, of such quantities as the infra-correlations we considered here. This is not the only subject we considered that might deserve further investigation, possibly for a more general application than the context of this work. Let us mention, for instance, the methods for finding regular and realistic forms for correlations easy to modify and control or the alternative techniques for missing or misaligned data in the swaption market. As far as the specific subject of cascade calibration is concerned, notice that this method, inducing a direct (one-to-one in the basic form) correspondence between model parameters and swaption volatilities, might prove to be particularly useful in calculating sensitivities with respect to specific swaptions, and in general for hedging purposes. This interesting opportunity has not been addressed in this work.

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