Credit and Default Modeling

UNIT 4

MULTI NAME CREDIT DERIVATIVES

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UNIT 4. Multi Name Credit Derivatives

In this unit we discuss financial payoffs depending on more than one underlying reference credit. The products we consider are

- First to default;
- n-th to default, Last to default;
- CDS indices;
- CDO tranches;
- CDO squared tranches.
- Leveraged super senior (LSS) CDO tranches;
- Standard Indices: DJ-i-Traxx and related tranches.

First to default

τ_i: default time of reference credit i; τ^i: default time of the i-th name that defaults.

τ^i depends on the trajectories. If in a trajectory ω first defaults name 3, second name 5 and third name 2, we have

$$\tau^1(\omega) = \tau_3(\omega), \quad \tau^2(\omega) = \tau_5(\omega), \quad \tau^3(\omega) = \tau_2(\omega).$$

In a different trajectory the order can be different.

A first to default (FtD) is similar to a CDS on a single name but this time the default that calls for the payment of protection and for the end of the premium leg is the first default in a basket of reference credits $1, 2, 3, \ldots, N$.

First to default

Two companies "A" (Protection buyer) and "B" (Prot seller) agree on the following.

If the first defaulting company among 1,2,...,N (Reference Credits) defaults at time $\tau^1(\omega) = \tau_{i_1(\omega)}(\omega)$, with $T_a < \tau^1 < T_b$, "B" pays to "A" $\text{L}_{\text{GD}}{}_{i_1}$. In turn, "A" pays to "B" a rate R at $T_{a+1}, \ldots, T_b$ or until default τ^1. Set $\alpha_i = T_i - T_{i-1}$ and $T_0 = 0$.

Protection Seller B	$\rightarrow$	$\text{L}_{\text{GD}}{}_{i_1}$ at 1st default $\tau^1 = \tau_{i_1}$ if in $[T_a, T_b]$	$\rightarrow$	Protection Buyer A
	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until 1st default τ^1	$\leftarrow$	

(protection leg and premium leg respectively).

The cash amount $\text{L}_{\text{GD}}{}_{i_1}$ is a *protection* for "A" in case the first name defaults before T_b and is name i_1. Typically $\text{L}_{\text{GD}} = 1 - \text{R}_{\text{EC}}$. Notice that i_1 is a random variable.

In standard FtD, written by sectors, all names have the same REC (0.5 for financial and 0.4 for telecoms for example).

First to default: Basic concepts

Usually the FtD's are composed by a small number of entities: $N = 5$ or 10.

Again, a FtD works similarly to the standard CDS, but the protection is paid against the first reference entity in the basket experiencing default: the protection seller assumes more risk with respect to selling protection on any single name in the basket.

This fact leads in turn to higher premium rates R paid from the protection buyer.

The protection seller is attracted by the leverage. In case of a default, the protection seller is due to make a single payment relative to a single reference entity, as in single name CDS, but it receives a larger rate before default due to the higher likelihood of default.

From the protection buyer viewpoint, a FtD is seen as a lower cost method of partially hedging multiple credits. However the buyer keeps the risk of multiple defaults.

First to default: Basic concepts

The premium paid for protection is essentially the premium paid for a single name protection **plus** a part due to the likelihood of multiple defaults, that depends on correlation.

Intuitively, in case of perfect dependence, i.e. all reference credits defaulting together, the premium paid for the FtD basket would be equal to the premium paid for the riskiest name. This can be seen also formally.

The correlation is usually meant not directly on the default times but on the variables ξ triggering default in the reduced form model; we will go deeply into this topic later, when discussing copula functions. Now we only give a feeling for this.

First to default: Basic concepts

Indeed, assume positive deterministic intensities γ_i, cumulated intensities Γ_i and defaults of each name i, τ_i, as first jump of a related Poisson process:

$$\tau_1 = \Gamma_1^{-1}(\xi_1), \ldots, \tau_N = \Gamma_N^{-1}(\xi_N)$$

If the ξ are all equal, the smallest τ_i (i.e. the first default) is the one corresponding to the largest Γ_i, i.e. to the riskiest single name. Then the related premium rate R will be the rate of the corresponding CDS

At the same time the premium paid for a FtD has to be lower than the sum of the single different premia of the basket components:

The sum of the CDS rates R_i corresponds to protection against all of the names up to time τ. As such, this includes in particular protection against the first name, and then the premium rate of the sum of CDS has to be larger than the premium rate in the FtD.

First to default, k-th to default

Formally we may write the (Running) FtD discounted value to "B" at time $t < T_a$ as

$$\Pi\text{RFtD}_{a,b}(t) := D(t,\tau^1)(\tau^1 - T_{\beta(\tau^1)-1})R\mathbf{1}_{\{T_a<\tau^1<T_b\}} +$$

$$+ \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau^1>T_i\}} - \mathbf{1}_{\{T_a<\tau^1\le T_b\}} D(t,\tau^1)\ \text{L}\text{GD}_{i_1}$$

The price of this product can be computed by risk neutral expectation. $D(t,T)$ is the discount factor between t and T (most of times assumed deterministic).

Similarly to the 1st to default, we have contracts such as 2nd to default, k-th to default, last to default. The protection in the generic k-th to default is paid in correspondence of the $k-th$ default τ^k among the names in the basket if this is in $[T_a, T_b]$.

Just substitute τ^1 with τ^k above and i_1 with i_k.

First to default, k-th to default

The initial fair R of the k-th to default, denoted R^k, can be computed as follows

$$R^k_{a,b} = \frac{\mathbb{E}_t[\mathrm{L}_{\mathrm{GD}}{}_{i_k} D(t,\tau^k)\mathbf{1}_{\{T_a<\tau^k\le T_b\}}]}{\mathbb{E}_t[D(t,\tau^k)(\tau^k - T_{\beta(\tau^k)-1})\mathbf{1}_{\{T_a<\tau^k<T_b\}}] + \sum_{i=a+1}^{b}\mathbb{E}_t[D(t,T_i)\alpha_i\mathbf{1}_{\{\tau^k>T_i\}}]}$$

We need the Monte Carlo simulation in general, except possibly in case $k = 1$ or $k = N$, where some tricks are available under some particular models;

In general we may be able to avoid simulation also for generic k-th to default under some approximations and homogeneity assumptions on recovery rates. More on this later.

Index CDS's (iTraxx, CDX...)

Given a pool of names $1, 2, \ldots, M$, typically $M = 125$, each with notional $1/M$, so that the pool has total notional 1, the index default leg pays to the protection buyer the loss increment occurring each time one or more names default, until final maturity $T = T_b$ arrives or until all the names in the pool have defaulted.

$$\text{Loss}(T) = \frac{1}{M}\sum_{i=1}^{M}(1-\text{Rec}_i)1_{\{\tau_i \le T\}}, \quad \text{Outno}(T) = \frac{1}{M}\sum_{i=1}^{M}1_{\{\tau_i > T\}} = 1 - \frac{\text{n. of defaults by } T}{M}$$

In exchange a periodic premium rate (or "spread") R is paid from the protection buyer to the protection seller, until final maturity T_b. This is computed on a notional that decreases each time a name in the pool defaults, and decreases of an amount corresponding to the notional of that name. Important: The whole notional, **irrespective of the recovery**.

Prot Seller	$\rightarrow$	Loss increment $d\text{Loss}(t)$ at all $t \in [T_0, T_b]$	$\rightarrow$	Prot Buyer
	$\leftarrow$	rate R at $T_1, \ldots, T_b$ on OutstandingNotional	$\leftarrow$	

Index CDS's (iTraxx, CDX...)

The price of the two legs of the index at time 0 is given as follows:

$$\text{PriceDefaultLeg}(0) = \mathbb{E}\left[\int_0^T D(0,u)d\text{Loss}(u)\right]$$

$$\text{PricePremiumLeg}(0) = R_{\text{index}}\mathbb{E}\left[\sum_{i=1}^{n} \alpha_i D(0,T_i)\,\text{OutNo}_{\text{index}}(T_i)\right].$$

where $\alpha_i = T_i - T_{i-1}$ is the year fraction.

One should not be confused by the integral, the loss $\text{Loss}(t)$ changes with discrete jumps each time one or more names in the pool default.

Index CDS's (iTraxx, CDX...)

$$\text{PricePremiumLeg}(0) = R_{\text{index}}\mathbb{E}\left[\sum_{i=1}^{n} \alpha_i D(0, T_i) \,\text{OutNo}_{\text{index}}(T_i)\right].$$

The quantity

$$\text{DV01}_{\text{index}} = \text{Annuity}_{\text{index}} = \mathbb{E}\left[\sum_{i=1}^{n} \alpha_i D(0, T_i) \,\text{OutNo}_{\text{index}}(T_i)\right]$$

represents the value of the premium leg corresponding to a unit basis point spread R_{index}, or also the change in the premium leg when the index spread changes by one basis point (keeping everything else fixed).

It is often called the (defaultable) annuity of the index

Index CDS's (iTraxx, CDX...)

Prot	$\rightarrow$	Loss increment $d\text{Loss}(t)$ at all $t \in [T_0, T_b]$	$\rightarrow$	Prot
Seller	$\leftarrow$	rate R at $T_1, \ldots, T_b$ on OutstandingNotional	$\leftarrow$	Buyer

$$\text{OUTNO}_{\text{index}}(t) = 1 - (\text{number of defaults by } t)/\text{number of names in the pool}$$

Differently from what will happen with the tranches, **the recovery is not considered when computing the outstanding notional**. Thus the index contains information both on the actual loss (including recoveries) and on the number of defaults alone.

The forward index spread is the R_{index} that sets the 2 legs to the same value at time 0

$$R_{\text{index}}(0) = \frac{\mathbb{E}\left[\int_0^T D(0,u)d\text{Loss}(u)\right]}{\mathbb{E}\left[\sum_{i=1}^n \alpha_i D(0,T_i)\,\text{OUTNO}_{\text{index}}(T_i)\right]}$$

The market quotes the value of $R_{\text{index}}(0)$ at specific dates $t = 0$ that, for different maturities T, balances the two legs.

CDO tranches

Synthetic CDO tranches with maturity T are obtained by "tranching" the loss "$\text{Loss}(t)$" of the index pool. The tranched loss at points A and B is

$$\text{Loss}^{tr}_{A,B}(t) := \frac{1}{B-A}\left[(\text{Loss}(t) - A)1_{\{A<\text{Loss}(t)\le B\}} + (B-A)1_{\{\text{Loss}(t)>B\}}\right].$$

The contract consists of two legs, the **default leg** and the **premium leg**.

Default Leg. Once enough names have defaulted and the loss has reached A, the count starts. Each time the loss increases the corresponding loss change re-scaled by the tranche thickness $B - A$ is paid to the protection buyer, until maturity arrives or until the total pool loss exceeds B, in which case the payments stop.

The discounted **default leg** can then be written as

$$\text{DefLeg}_{A,B}(0) = \int_0^T D(0,t)d\text{Loss}^{tr}_{A,B}(t)$$

CDO's tranches

Premium leg. As usual, in exchange for the loss payments, a premium rate $R_{0,T}^{A,B}(0)$, fixed at time $T_0 = 0$, is paid periodically, say at times $T_1, T_2, \ldots, T_b = T$, from the protection buyer to the protection seller. Part of the premium can be paid at time $T_0 = 0$ as an upfront $U_{0,T}^{A,B}(0)$. This time the rate is paid on the "survived" positive notional at the relevant payment time and not on the whole notional. **This notional decreases of the same amounts as the tranched loss increases, taking into account the recovery, differently from the index**. This notional is $\textsc{Outno}_{A,B}^{tr}(t) = 1 - \textsc{Loss}_{A,B}^{tr}(t)$.

The **premium leg** payoff is

$$\textsc{PrLeg}(0) = U_{0,T}^{A,B}(0) + \sum_{i=1}^{b} D(0, T_i) R_{0,T}^{A,B}(0) \int_{T_{i-1}}^{T_i} \textsc{Outno}_{A,B}^{tr}(t) dt.$$

If we assume that the payments are made on the notional remaining at each payment date T_i, the premium leg can be written by replacing the integral with $\alpha_i(1 - \textsc{Loss}_{A,B}^{tr}(T_i))$.

CDO's tranches

Prot Seller	$\rightarrow$	Tranched loss increment $d\text{Loss}^{tr}_{A,B}(t)$ at all $t \in [T_0, T_b]$	$\rightarrow$	Prot Buyer
	$\leftarrow$	(upfront U at T_0, and) rate R at $T_1, \ldots, T_b$ on OutNo	$\leftarrow$	

The premium rate $R^{A,B}_{0,T}(0)$ that makes the contract fair (i.e. zero valued) at inception can be derived by equating the risk neutral expectation of the two legs and solving in R:

$$R^{A,B}_{0,T}(0) = \frac{\mathbb{E}_0\left[\int_0^T D(0,t) d\text{Loss}^{tr}_{A,B}(t)\right] - U^{A,B}_{0,T}(0)}{\mathbb{E}_0\left[\sum_{i=1}^b \alpha_i D(0,T_i)(1 - \text{Loss}^{tr}_{A,B}(T_i))\right]}$$

and the above expression can be easily recast in terms of the upfront premium U for tranches that are quoted in terms of upfront fees. Again, **differently from the index, the tranche has the same loss in numerator and denominator**. Thus tranche data alone (without index) do not allow for splitting the information on number of defaulted names and actual loss, or, they do not allow to single out the **recovery** from the loss.

CDO's tranches

The market quotes either the periodic premiums rate $R_{0,T}^{A,B}(0)$ of these tranches or their upfront premium rate $U_{0,T}^{A,B}(0)$ for maturities $T = 3y, 5y, 7y, 10y$ and standard attachment points A, B, on standardized pools of names.

Tranches with low detachment points $(B \leq 3\%)$ are usually quoted in terms of the upfront premium, while tranches with higher detachment points are quoted in terms of the periodic premium.

Information contained in CDO quotes

Recall the market quoted spreads for indices and tranches on standardized pools:

$$R_{\text{index}}(0) = \frac{\mathbb{E}_0\left[\int_0^T D(0,u)d\text{Loss}(u)\right]}{\mathbb{E}_0\left[\sum_{i=1}^n \alpha_i D(0,T_i)\ \text{Outno}_{\text{index}}(T_i)\right]}$$

$$\text{Outno}_{\text{index}}(t) = 1 - (\text{number of defaults by } t)/M$$

$$R_{0,T}^{A,B}(0) = \frac{\mathbb{E}_0\left[\int_0^T D(0,t)d\text{Loss}_{A,B}^{tr}(t)\right] - U_{0,T}^{A,B}(0)}{\mathbb{E}_0\left[\sum_{i=1}^b \alpha_i D(0,T_i)(1 - \text{Loss}_{A,B}^{tr}(T_i))\right]}$$

If these spreads R are the only implied correlation we have in the market, **we see that the only information we can infer are “expected losses”, “expected tranche losses” and “expected number of defaults”.**

CDS Indices: DJ-iTRAXX in detail

DJ-iTRAXX is a family of CDS indices, which spans the main credit market in Europe.

This family was created with the purpose to standardize market quotes, and also to create a reference liquid multi-name credit derivative. These indices constitute now the most liquid quotations in the credit-derivatives market.

Since the quotation paradigm is standardized (we see the details below) these reference quotes are practically the only safe source of market cross-sectional default correlation.

There are also indices for different areas. We can find indices relative to Europe, the US (CDX), Japan, Asia, Australia, high yields and emerging markets.

We will concentrate on the European index: The DJ iTraxx Europe.

CDS Indices: DJ-iTRAXX Europe

These credit derivatives indices provide exposure to the high grade credit markets, and in particular to the most liquid reference entities.

In Europe (and also in the US), these indices also provide exposure to the main sub-sectors of the high grade corporate bond markets, and also to high yield.

The credit indices are constructed in order to provide exposure to the most liquid segments of the credit markets. This is achieved by **selecting the most liquid CDS** in the market and **equally weighting them in the index**

Each index is subject to regular rebalancing every 6 months in March and September. Rebalancing follows the same rules as the initial composition of the indices.

CDS Indices: DJ-iTRAXX Europe

DJ iTraxx Europe comprises a variety of indices. The main four are:

- **DJ iTraxx Benchmark ("main index")** - Top 125 European names in terms of the CDS volumes traded by the market makers in the past six months. Sectorial diversification; 3,5,7 and 10y maturities;
- **DJ iTraxx HiVol Index** - This index is a subset of DJ iTraxx Benchmark (30 names), but the reference entities are supposed to be the most volatile names in the index. They are those with the widest default swap spread on a certain date; 3,5,7 and 10y maturities;
- **DJ iTraxx Crossover** - 45 names as selected by a dealers' poll. Non-financial European names with rating no better than Baa3 or BBB- and on Negative Watch or worse; 5y and 10y maturities.

All indices roll every six months, in **March** and in **September**.

CDS Indices: DJ-iTRAXX Europe

We concentrate on the most important index, i.e. the DJ iTraxx main index.

We have seen before that when a **credit event** happens in one of the names in the credit index, the seller of protection pays an amount equal to the Loss Given Default (LGD) of that particular name, proportionally to the weight of the name in the pool (the names are equally weighted).

On the other side the protection buyer still pays the index premium rate, but on the decreased notional corresponding to the names still "alive".

CDS Indices: DJ-iTRAXX Europe

DJ iTraxx Europe can easily be used as a simple and cheap instrument to trade the general direction of credit spreads. It has a number of advantages:

- Immediate diversification. DJ iTraxx Europe enables the investor to gain immediate diversification in a single liquid transaction.
- Accurate market tracking. The inclusion of only the most liquid names and the fact that these are updated every six months, ensures that DJ iTraxx Europe accurately reflects the composition of the European credit market.
- Low bid/offer spread. DJ iTraxx Europe has the lowest bid/offer spread of any instrument in the EUR investment grade market. At a bid/offer spread about 1/2bp, transactions are below the cost of the cheapest benchmark instruments.
- High liquidity. The large number of market makers ensures that investors can trade large sizes without affecting the market.

CDS Indices: Weighted- vs uniform- average DJ-iTRAXX spread

Consider all the names in the i-TRAXX portfolios. Take the 5y maturity. Consider an index replicating portfolio of CDS on each of the 125 names of i-traxx and assume premium rates (or spreads) R^i for each name to be paid at times $T_1, T_2, \ldots, T_b$.

The protection leg of this portfolio is the sum of the protection legs of the 125 individual CDS. Each protection leg is equal to the premium leg of the same CDS, given that at inception the CDS price is null. Then the protection leg for each name i is

$$\text{ProtLeg}_i = R^i_{0,5y} \sum_{j=1}^{b} \alpha_j \bar{P}_i(0, T_j) = R^i_{0,5y} A_i$$

where $\bar{P}_i(0, T_j)$ is the corporate zero coupon bond for name i and maturity T_j and A_i is the related annuity (or DV01). As usual α_j is the year fract between T_{j-1} and T_j.

CDS Indices: Theoretical vs actual DJ-iTRAXX spread

If we look for a single premium rate $\bar{R}$ to be paid at periods $1, 2, \ldots, b$, in all the premium legs of the replicating portfolio, that balances all the protection legs, we have to solve the equation between the related premium leg and the total protection leg

$$\bar{R} \sum_{i=1}^{125} \sum_{j=1}^{b} \alpha_j \bar{P}_i(0, T_j) = \sum_{i=1}^{125} \mathsf{ProtLeg}_i$$

or, given the above equality

$$\bar{R} \sum_{i=1}^{125} A_i = \sum_{i=1}^{125} R^i_{0,5y} A_i \qquad \text{so that}$$

$$\boxed{\bar{R} = \sum_{i=1}^{125} w_i R^i_{0,5y}} \;, \quad w_i = \frac{A_i}{\sum_{k=1}^{125} A_k}$$

CDS Indices: Theoretical vs actual DJ-iTRAXX spread

$$\boxed{\bar{R} = \sum_{i=1}^{125} w_i R^i_{0,5y}}\ , \quad w_i = \frac{A_i}{\sum_{k=1}^{125} A_k}$$

This would be the correct index premium rate (or spread) obtained through the replicating portfolio; If we computed instead, as one is tempted to do for simplicity,

$$\boxed{\hat{R} = \sum_{i=1}^{125} \frac{1}{125} R^i_{0,5y}}$$

this would be the correct spread only if defaults never occured. The discrepancy $\bar{R} - \hat{R}$ depends on the single name CDS survival probabilities and it is larger when single name CDS spreads are more dispersed. The default probabilities embedded in the $\bar{P}_i$'s are of course stripped from CDS; They can be stripped not only from the 5y CDS premium rates $R^i_{0,5y}$ of the different names but also from $R^i_{0,1y}$, $R^i_{0,3y}$ etc.

CDS Indices: DJ-iTRAXX tranches

As we have seen before for CDO's the index can be traded also in terms of tranches.

The difference with earlier general CDO's is that now tranches are standardized.

For the DJ-iTraxx Europe, the traded tranches are: An **Equity tranche**, responsible for all losses between $A = 0\%$ and $B = 3\%$, then other **mezzanine** and **senior** tranches covering

$$A - B: \quad 3\% - 6\%,\ 6\% - 9\%,\ 9\% - 12\%,\ 12\% - 22\%.$$

For the main US index, the DJ CDX NA the tranche sizes are different:

$$0\% - 3\%,\ 3\% - 7\%,\ 7\% - 10\%,\ 10\% - 15\%\ 15\% - 30\%.$$

CDS Indices: DJ-iTRAXX tranches

Example: Investor sells EUR 10mn protection on the 3%-6% tranche. We assume a credit spread of 135bp. Therefore, the market maker pays the investor 135bp per annum quarterly on a notional amount of EUR 10mn. We assume REC $= 40\%$ for all names.

- Each single name in the portfolio has a credit position in the index of $1/125 = 0.8\%$ and participates to the aggregate loss in terms of 0.8% x LGD = 0.8% x 0.6 = 0.48%.
- This means that each default corresponds to a loss of 0.48% in the global portfolio.
- After 6 defaults, the total loss in the portfolio is EUR 0.48% x 6 = 2.88%, and the tranche buyer is still protected.
- When the 7th name in the pool defaults the total loss amounts to 3.36% and the lower attachment point of the tranche is reached.
- To compute the loss of the tranche we have to normalize the total loss with respect to the tranche size: The net loss in the tranche is then (3.36%-3%)/3% x 10mn = EUR 1.2mn which is immediately paid by the protection seller.

CDS Indices: DJ-iTRAXX tranches

Example (Continued)

- The notional amount on which the premium is paid reduces to 10mn - 1.2mn = EUR 8.8mn, and the investor receives every month a premium of 135bp on EUR 8.8mn until maturity or until the next default.
- Each following default leads to change in the tranche loss (paid by the protection seller) of 0.48%/3% x 10mn = EUR 1.6mn, and the tranche notional decreases correspondingly.
- After the 13th default the total loss exceeds 6% (13 x 0.48% = 6.24%) and the tranche is completely wiped out.
- In this case one last payment is made of (6%-5.76%)/3% x 10mn = EUR 0.8mn to the protection buyer, which in turn stops paying the premium since the outstanding notional has reduced to zero.

CDS Indices: DJ-iTRAXX tranches

An equity tranche buyer suffers of every default in the portfolio, which leads to a decrease of tranche notional on which the periodic premium is paid and conversely to contingent protection payments.

On the other side, the buyer of more senior tranches is more protected against few defaults: Indeed these tranches are affected only in case of a large number of defaults.

This difference leads to different premia paid to buy protection. It is natural to see that the periodic premium paid to buy protection is inversely proportional to the tranche seniority: the larger A and B, the smaller $R^{A,B}$.

A technical note: The premium for the equity tranche is usually very large, so it is market practice to pay it as a fixed running premium $R = 500$bps plus an upfront payment U (still computed so that the total value is zero at inception).

CDS Indices: DJ-iTRAXX tranches. Quotes examples

d1: 03-Oct-05; d2 : 23-May-06; quotes in bps;
0-3% equity tranche is quoted upfront + 500bps running

	Rindex3y	**Upfront0-3(3Y)**	**R3-6(3Y)**	**R6-9(3Y)**
d1	21.25	0.0475	20	7
d2	17.5	0.005	3.5	1.125

	Rindex5y	**Up0-3(5y)**	**R3-6(5y)**	**R6-9**	**R9-12**	**R12-22**	**R22-100**
d1	36	0.2775	91	27.5	11.5	6.5	2.42301
d2	32	0.24175	69.25	21	9.25	4.25	1.55

	Ridx10y	**Up0-3(10y)**	**R3-6(10y)**	**R6-9**	**R9-12**	**R12-22**	**R22-100**
d1	55.75	0.565	482.5	101.5	48.5	20.5	9.83936
d2	53.5	0.515	552.5	131	62	23.5	4.5

CDO Squared

As CDOs have become more popular, issuers have started looking for new assets against which to collateralize them, including Asset Backed Securities (ABS) and even other CDOs. This explains the birth of CDOs of ABS, CDOs of CDOs and also CDOs of a mixture of ABS and CDOs. These products are usually called CDOs squared. Due to leverage effects, these products create different risk profiles from the regular synthetic CDO, resulting in higher spreads for the investors.

Typically these products are composed both by ABS and CDOs and, as a first proxy, the ABS can be considered default free, leaving all the default risk to the underlying CDO tranches. For example we can think of a portfolio composed by 80% of ABS and the remaining by mezzanine tranches of six CDOs.

CDO Squared

The attachment points of the mezzanine tranches can be different and the underlying portfolios are also different for each CDO. Obviously, given the limited number of credit entities present in the market, a certain overlapping of names across basic CDO's to be retranched is inevitable. The whole portfolio is then re-tranched and sold to the investors.

The basic idea is quite simple and it is similar to the regular CDO case: The fundamental variable is always the loss. Since we assume that the ABS are default free, the only losses can be given by the underlying CDO tranches. But here we have a sort of double protection: In the regular CDO each default impacts on some tranche (for example the first default attacks the equity tranche). Here before impacting the squared tranche, the underlying tranches must be impacted.

CDO Squared

For example if only one name defaults, we could have no effect on the underlying mezzanines and hence no effect on the squared tranches. On the other side, when the mezzanine tranches are impacted, every successive default impacts deeply on the mezzanine tranche and hence on the squared tranches. This is why we have higher spreads for the same risk profile when compared to regular CDO tranches.

Furthermore it is clear that the overlapping effect is very relevant, since the default of one single name which is present in many CDOs could lead to great portfolio losses.

CDO Squared

Summarizing, CDO squared are CDO of CDO's (hence the name) in the following sense.

As we have seen above, a CDO protects against the tranched loss of the pool.

Now suppose for a moment that we take a set of $1, 2, \dots, M$ tranches of different "basic" CDO's as underlyings of a global CDO. In principle the underlying CDO's can be written on different portfolios and can have different attachment points.

Here, for simplicity, we assume the basic CDO's to have the same maturity T_b, the same attachment points A, B and different underlying portfolios built using an equal number of components.

CDO Squared

Now add the tranched losses $(\text{Loss}^{tr}_{A,B})^{(1)}, (\text{Loss}^{tr}_{A,B})^{(2)}, \ldots, (\text{Loss}^{tr}_{A,B})^{(M)}$ of all these basic CDO's $1, 2, \ldots, M$...

...and tranche again the obtained total tranched losses $\mathcal{L}$ along two points X, Y!!

Roughly speaking, a CDO squared is a contract offering protection against this re-tranched loss of a pool of CDO tranches.

CDO Squared

Formally, define the total Loss of the basic CDO tranches

$$\mathcal{L}(t) := (\mathrm{Loss}^{tr}_{A,B})^{(1)} + (\mathrm{Loss}^{tr}_{A,B})^{(2)}(t) + \ldots + (\mathrm{Loss}^{tr}_{A,B})^{(M)}(t);$$

Notice that since each basic tranche loss $\mathrm{Loss}^{tr}_{A,B}(t)$ is always a number between 0 and 1, we have easily

$$\mathcal{L}(t) \in [0 \;\; M].$$

Now tranche again at $X, Y \in [0 \;\; M]$:

$$\mathcal{L}_{X,Y}(t) := \frac{1}{Y - X}\left[(\mathcal{L}(t) - X)1_{\{X<\mathcal{L}(t)\leq Y\}} + (Y - X)1_{\{\mathcal{L}(t)>Y\}}\right]$$

As for the basic CDO tranches, protection is offered against this loss in exchange for a flow of interest rate payments R occurring at given times $T_1, T_2, \ldots, T_b$ on the total notional minus the losses incurred up to that point.

CDO Squared

Also in CDO squared tranches one is interested in the premium rate $(R_{0,T}^{X,Y})^{(2)}(0)$ that makes the contract fair (i.e. zero valued) at inception.

We may write the CDO squared discounted payoff to the protection buyer as

$$\int_0^T D(0,t)d\mathcal{L}_{X,Y}(t) - \sum_{i=1}^{n} (R_{0,T}^{X,Y})^{(2)}(0)\alpha_i D(0,T_i)(1-\mathcal{L}_{X,Y}(T_i))$$

As usual, the fair premium rate at inception is defined as the one that sets to zero the risk neutral price of the CDO squared:

$$(R_{0,T}^{X,Y})^{(2)}(0) = \frac{\mathbb{E}_0\Big[\int_0^T D(0,t)d\mathcal{L}_{X,Y}(t)\Big]}{\mathbb{E}\Big[\sum_{i=1}^n \alpha_i D(0,T_i)(1-\mathcal{L}_{X,Y}(T_i))\Big]}$$

CDO Squared

The difference with basic CDO's is not in the shape of the payoff in terms of a given loss and premium rates (they are identical) but is in the the underlying loss to be tranched.

$\mathcal{L}_{X,Y}(t)$ here contains much more structure than the loss tranched in the basic CDO tranches.

For example, the expected loss $\mathbb{E}[\mathcal{L}]$ depends on the correlation structure while the expected loss $\mathbb{E}[\text{Loss}]$ of the basic pool of names (CDO's) does not.

Leveraged Super Senior (LSS) CDO tranches

These are products that try and render senior tranches attractive to investors.

Usually, in a CDO tranche, if the attachment and detachment points A and B are very far away from 0 ("super-senior tranche"), the probability that the loss hits the $[A, B]$ interval is very small.

Consequently, the premium rate $R_{0,T}^{A,B}(0)$ that is fair to exchange against protection for the $[A, B]$ tranche is very low.

This discourages investors from selling protection on plain super-senior tranches.

Leveraged Super Senior (LSS) CDO tranches

In other terms, to obtain an interesting periodic premium $R\ N_{\text{BIG}}$, proportional to the initial notional N_{BIG}, the investor would need to sell protection on a huge notional N_{BIG}.

This would expose the protection seller to possibly huge protection payments.

To avoid this, a trigger event leading to unwinding is introduced. Ideally, this trigger should lead to the contract unwinding when the protection payments start getting severe.

The unwind is capped at an amount N_{SMALL}, so that the protection seller (investor) risks a loss payment on a notional of at most N_{SMALL}.

Leveraged Super Senior (LSS) CDO tranches

More formally, consider a senior tranche $[A, B]$.

Assume we are the investor (protection seller) and we wish a high R by risking only on a notional $N_{\text{SMALL}} < N_{\text{BIG}}$, smaller than the whole super-senior tranche size.

There is a trigger event based on the underlying portfolio loss and possibly on the average premium rate (spread) of the portfolio,

$$\text{Loss}(t), \quad \hat{R}_{t,T} = \frac{1}{N} \sum_{i=1}^{N} R^i_{t,T} \;(\text{ or } R_{\text{index}} \text{ if available}),$$

where $R^i_{t,T}$ is the CDS rate (or spread) for name i that balances protection in the time interval $[t, T]$. The trigger event can be for example the first instant t when:

The loss $\text{Loss}(t)$ exceeds a pre-specified safety level $K(t)$;

Leveraged Super Senior (LSS) CDO tranches

A more complex trigger event can be the first instant t when:

The average spread $\hat{R}_{t,T}$ exceeds the spread given in a trigger matrix as a function of time to maturity $T - t$ and of the loss rate $l_t = (\text{Loss}(t) - \text{Loss}(0))/\text{Loss}(0)$

$l_t \downarrow$ / $T - t \rightarrow$	$2y$	1.5	1	0.5
0%	500	650	900	1800
2%	390	510	750	1500
4%	...	...	...	....
8%	...	...	...	....
10%	90	110	180	350

Leveraged Super Senior (LSS) CDO tranches

Let us call τ the time of the trigger event. The LSS can be schematized as follows:

Prot Seller	$\rightarrow$	$N_{\mathsf{BIG}}\ d\mathsf{Loss}^{tr}_{A,B}(t)$ until the earliest of τ and T_b	$\rightarrow$	Prot Buyer
	$\rightarrow$	unwind: $\min(N_{\mathsf{BIG}}\mathsf{Tranche}_{A,B}(\tau), N_{\mathsf{SMALL}})$at τ if before T_b	$\rightarrow$	
	$\leftarrow$	R at $T_1, \ldots, T_b$ on OutNo N_{BIG}, but stop at τ	$\leftarrow$	

where $N_{\mathsf{BIG}}\ \mathsf{Tranche}_{A,B}(\tau)$ is the value (mark to market) at τ of the $[A, B]$ (super-senior) tranche to the protection buyer (protection leg - premium leg) on notional N_{BIG}.

The protection seller/investor pays protection only on a small N_{SMALL} and receives the tranche premium on a large N_{BIG}.

Clearly the optional part triggered by τ has a price that has to be included in the contract. Valuation of this component is a nontrivial pricing problem.

Leveraged Super Senior tranches: Market features

LSS are One of the latest innovations in the synthetic CDO market in 2005

The low absolute return of unleveraged super-senior tranches prevents protection sellers from acting in the super-senior market.

Adding leverage as above makes it possible to achieve a higher coupon on the cash invested, encouraging a larger participation by traditional senior and mezzanine investors.

SS tranches have tightened considerably in 2005, particularly 7 and 10 year tranches

7y and 10y Mezzanine and equity tranches have widened instead

This happened because of the tight levels of 5y mezzanine tranches

Leveraged Super Senior tranches: Market features

With 5y mezzanine tranches tight, investors had two possibilities: longer maturities or leveraging more senior tranches.

This led to both 7- and 10-year mezzanine transactions and LSS transactions.

Conclusions

We have seen that in general in multi name products the interdependence among the different components in the portfolio is quite relevant.

For example in k-th to default products the correlation among default times is relevant to evaluate the distribution of the τ's.

Analogously the correlation has a deep impact on the distribution of the portfolio loss, which is the main variable to take into account when pricing CDO's and iTraxx tranches.

This topic will be addressed later in the discussion, when we will introduce the concepts of default correlation and copula functions.