UNIT 3

SINGLE NAME MODELS: STRUCTURAL (FIRM VALUE)

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UNIT 3. Single Name Credit Derivatives: Structural

- Basic ideas of Structural Models: comparison with Intensity Models (Pros/Cons);
- Merton's model: Basic capital structure for the firm value and very simple definition of default time;
- Black and Cox model and default barrier models in general: More realistic definition of the default event;
- Analytically Tractable First Passage model (AT1P);
- Credit Default Swap calibration with the AT1P model: The Parmalat example;
- Extension: Random Barrier and Random Volatility. The Scenario Volatility and Barrier Analytically Tractable First Passage model;
- Using Structural Models to price counterparty risk and hybrid equity/credit products;
- Hints at Equity Default Swaps pricing;
- Survey on other Structural Models (CreditGrades model).
- References

The Basic Idea of Structural Models

The stylized structure of the firm economy is modeled through:

- $V(t)$: **stochastic process for the value of the firm**
- $t \mapsto H(t)$ **default barrier representing the debt** of the firm and safety covenants.
- τ: **The default time is the first time instant where the value of the firm V touches the safety barrier H.**

The basic idea is that if the firm value goes below the safety level, then the firm is deemed to be no longer able to pay its debt and is forced into bankruptcy.

Default is induced by observable market information (the value of the firm V).

Important difference with Intensity Models: In basic structural models there is nothing external to the basic market information in the default process. Default is induced by a completely observable variable, the value of the firm.

For these models we have $\mathcal{G}_t = \mathcal{F}_t$, i.e. history on past and present default plus the basic market information coincides with the history of the basic market information itself.

Intensity and Structural Models: Different Uses

At this stage it is not possible to say that intensity models are better than structural models or vice versa. The two kind of models are both useful and both needed, and are to be used for different products.

Intensity Models:

- Intensity models offer parallels with interest rate models, and are thus more suited to model credit spreads;
- They are easier to calibrate to corporate bond or Credit Default Swap market information;
- More suited to refined relative value pricing (CDS options etc);
- In cases with stochastic intensity the extension to multiname situations can be difficult (First to default, CDO's, etc) and even the Monte Carlo method presents some problems;
- Calibration of the "default correlation" component (jump-terms Copula function) among different names is not clear and often based on debatable links with the equity market.

Intensity and Structural Models: Different Uses

Structural models

- Structural models are easier to use in situations where we need to model also equity variables taking into account correlation;
- In their basic formulation they are more suited to "fundamental pricing" that to refined relative value pricing, even if they are often used for the latter through some tricks;
- Cases include equity return swaps with counterparty risk, total rate of return swaps, and counterparty risk in any equity product, and Equity Default Swaps
- They are more difficult to calibrate with precision to Credit Default Swaps or Corporate Bonds data;
- They are more naturally extended to multi-name situation (no "out of the blue" copula) than stochastic intensity models;
- Different names default correlation has a much more grounded link with equity correlation than in intensity models; In principle it would suffice to estimate historical correlation of equity returns.

Structural Models: Merton's Model

The first Structural Model is due to Merton (1974). The **value of the firm V is assumed to be a tradable asset and to follow a standard Geometric Brownian Motion**. Under the risk neutral measure:

$$dV(t) = (r - k)V(t)dt + \sigma V(t)dW(t)$$

(r is the risk free rate, k is the payout ratio and σ is the volatility, all constant).

This dynamics is lognormal; Crouhy et al (2000) notice that "this assumption [lognormal V] is quite robust and, according to KMVs own empirical studies, actual data conform quite well to this hypothesis.".

V is seen as composed by the equity part S and the debt part D, so that at each point in time the following equivalence holds:

$$\text{Firm Value} = \text{Debt Value} + \text{Equity}, \quad V(t) = D(t) + S(t)$$

Structural Models: Merton's Model

The debt is simple: zero coupon debt with maturity $\bar{T}$ and face value L.

Default is linked to capability of the firm to pay back all the debt issued.

If at maturity $\bar{T}$ the firm value V is greater than L, then all the debt is paid back and the firm survives; if V is smaller than L then the company is not able to pay the bondholders and then there is the default. Analytically

$$\tau = \bar{T}\,\mathbf{1}_{\{V_{\bar{T}}<L\}} + \infty\,\mathbf{1}_{\{V_{\bar{T}}\geq L\}}$$

Default can happen only at the debt maturity $\bar{T}$. This is a quite restrictive assumption and we will see later in the discussion how it can be relaxed.

The value of the debt at maturity is hence $D_{\bar{T}} = \min(V_{\bar{T}}, L)$, from which, at $t < \bar{T}$

$$D_t = \mathbb{E}_t[D(t,\bar{T})\min(V_{\bar{T}}, L)] = \mathbb{E}_t[D(t,\bar{T})[V_{\bar{T}} - (V_{\bar{T}} - L)^+]] =$$

$$= \mathbb{E}_t[D(t,\bar{T})[L - (L - V_{\bar{T}})^+]] = P(t,\bar{T})L - \mathsf{Put}(t,\bar{T}; V_t, \sigma^2, L)$$

Merton's Structural Model. Linking Equity and Firm value

Since we assume $V(t) = D(t) + S(t)$, we simply have that the equity value is

$$\boxed{S(t)} = V(t) - D(t) = V(t) - P(t, \bar{T})L + \text{Put}(t, \bar{T}; V(t), \sigma^2, L)$$

$$= \boxed{\text{Call}(t, \bar{T}; V(t), \sigma^2, L)} = V(t)\Phi(d_1) - P(t, \bar{T})L\Phi(d_2) \quad (11)$$

where Φ is the normal cdf, $d_1 = \frac{\ln\left(\frac{V(t)}{L}\right) + \left(r - k + \frac{\sigma^2}{2}\right)(\bar{T} - t)}{\sigma\sqrt{\bar{T} - t}}$ and $d_2 = d_1 - \sigma\sqrt{\bar{T} - t}$

Then, as is well known, **in Merton's model the equity can be interpreted as a call option on the value of the firm**.

If we have estimates for r, k, σ, we can use this model to evaluate some particular payoff depending on V or to compute default probabilities. While r and k can be simple to estimate, σ is not, since the true process V is not really directly observable. A possible way to estimate σ is **to link it to the equity volatility**, available by market data.

Merton's Structural Model. Linking Equity and Firm value

To avoid confusion: σ_V is **the firm value volatility**, σ_S is the **equity volatility**.

We have seen the interpretation of equity as a call on V: $S(t) = \text{call}(t, V(t))$. Then, by differentiating and using Itô's formula, we obtain

$$dS(t) = d\text{call}(V(t)) = (\ldots)dt + \frac{\partial \text{call}}{\partial V}\sigma_V V(t) dW(t).$$

Comparing with an hypothetical dynamics $dS(t) = (r-q)S(t)dt + \sigma_S(t)S(t)dW(t)$ we immediately find the following important relation between σ_V and σ_S

$$\boxed{\sigma_S = \sigma_V \Delta_{\text{call}} \frac{V}{S}} \tag{12}$$

where $\Delta_{\text{call}} = \frac{\partial \text{call}}{\partial V} = \Phi(d_1)$ is the Delta greek of the call.

Merton's Structural Model: Linking Equity and Firm Value

Hull, Nelken and White (2004) use the implied volatilities of equity options to make statements on the capital structure of the firm and on the assets volatility σ_V.

Their reasoning: if equity is an option on V, equity options are options on options (i.e. compound options) on V for which there are analytical formulas due to Geske (1977).

To ease notation let us set ourselves in $t = 0$. Then the price of an **equity put** with strike K and expiry time $T^* < \bar{T}$ is

$$\text{PutGeske}(K) = LP(0,\bar{T})\Phi_2\left(-a_2, d_2; -\sqrt{\frac{T^*}{\bar{T}}}\right) - V_0\Phi_2\left(-a_1, d_1; -\sqrt{\frac{T^*}{\bar{T}}}\right) + KP(0,T^*)\Phi(-a_2)$$

where $a_1 = \left(\ln\left(\frac{V_0}{\hat{V}_{T^*}}\right) + \left(r - k + \frac{\sigma^2}{2}\right)T^*\right) / \left(\sigma\sqrt{T^*}\right)$, $a_2 = a_1 - \sigma\sqrt{T^*}$,
$\Phi_2()$ is the bivariate cumulative normal distribution and $\hat{V}_{T^*}$ is the critical asset value for which the equity value at time T^* equals K.

Merton's Structural Model: Linking Equity and Firm Value

An equity put can also be computed with standard BlackScholes formula, inconsistently with Merton's model (V and $S =$Call(t, V) cannot be both lognormal)

$$\text{PutBS}(0, T^*; S_0, \text{IVol}^2(K), K) = Ke^{-rT^*}\Phi(-d_2^*) - S_0\Phi(-d_1^*) \tag{13}$$

with IVol(K) the implied equity vol smile,

$$d_1^* = \left(\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\text{IVol}^2(K)}{2}\right) T^*\right) /\text{IVol}(K)\sqrt{T^*},\ d_2 = d_1 - \text{IVol}(K)\sqrt{T^*}.$$

Without entering the details we just mention the fact that if we equate this expression with the previous one for two quoted strikes

$$\text{PutGeske}(\sigma^2, L, K_1) = \text{PutBS}(\text{IVol}^2(K_1), L, K_1),$$

$$\text{PutGeske}(\sigma^2, L, K_2) = \text{PutBS}(\text{IVol}^2(K_2), L, K_2)$$

we can solve in σ and L.

Merton's Structural Model: Linking Equity and Firm Value

Hull, Nelken and White estimate the asset parameters. They use $T^* =$ two months $K_1 =$25-delta and $K_2 =$ 50-delta put volatilities (two relatively liquid options).

The two main advantages of this approach are that it is the **first attempt to fit a structural model to the implied volatility surface** and also it allows, by changing the debt maturity $\bar{T}$, to **find a term structure of default probabilities**.

This procedure uses **equity** (option) **data** to calibrate the firm value and barrier parameters σ and L. However, it ignores CDS data, the most liquid data one can find in single name credit. **Can we adapt the model to retrieve CDS data?**

In CDS contracts default may happen at any time before the final maturity. So we need a model where default is not restricted to a final maturity (as in Merton's). This leads us to introduce Black and Cox's model.

Structural Models: Black-Cox Model

Drawbacks of Merton's Model: default can happen only at the debt maturity $\bar{T}$.

Unsatisfactory: there could be scenarios in which default happens before $\bar{T}$, related to problems of optimal capital structure and stockholders decisions to reorganize the firm.

Black and Cox (1976) assume a barrier representing safety covenants for the firm. Default is triggered by the firm value V hitting this barrier from above. At default the firm reimburses the debt-holders. Let $H(t;\bar{T})$ be the barrier with time dependence on t and final zero coupon debt maturity $\bar{T}$.

Black and Cox assume a constant parameters Geometric Brownian Motion

$$dV(t) = (r-k)V(t)dt + \sigma_V V(t)dW(t)$$

and an exponential barrier (we omit the $\bar{T}$ dependence in H)

$$H(t) = \begin{cases} L & t = \bar{T} \\ Ke^{-\gamma(\bar{T}-t)} & t < \bar{T} \end{cases}$$

Structural Models: Black-Cox Model

$$H(t) = \begin{cases} L & t = \bar{T} \\ Ke^{-\gamma(\bar{T}-t)} & t < \bar{T} \end{cases},$$

γ and K are positive parameters. B&C assume also that $Ke^{-\gamma(\bar{T}-t)} < Le^{-r(\bar{T}-t)}$.

This second assumption means that the safety covenants are lower than the final debt present value, cutting some slack to the firm to recover stability and avoid default.

In this framework the default time τ is defined as

$$\tau = \inf\left\{t \in [0, \bar{T}] : \ V_t \leq H(t)\right\} \qquad (\inf \emptyset = \infty).$$

This is the reason why sometimes default barrier models are called first passage time models. If we set $\gamma = 0$ we have the particular case of a flat barrier.

If the dynamics parameters are constant, the default/survival probabilities can be directly computed (see Bielecki and Rutkowski (2001) for the derivation).

Black Cox: Barrier Options

Pretend for a moment that we are just doing derivatives pricing in an equity market with V as equity underlying.

There are many analytical formulas for barrier options pricing such as, for example, knock-in, knock-out and digital options.

One particular case is the down and out digital option or down and out bond (DOB), a contract paying 1 at T if, between the starting date of the contract and its maturity T, the underlying never touches the barrier $H(t)$ from above. Analytically

$$\bar{P}(0,T) = \text{DOB}(0,T) = \mathbb{E}\{D(0,T)\,\mathbf{1}_{\{\tau>T\}}\}$$

Since we will assume deterministic interest rates, in our case $D(t,T) = P(t,T)$, the zero coupon bond price at time t for maturity T. Then

$$\bar{P}(0,T) = P(0,T)\,\mathbb{E}\{\mathbf{1}_{\{\tau>T\}}\} = P(0,T)\,\mathbb{Q}\{\tau > T\}.$$

The last factor is the probability of never touching the barrier before T.

Black Cox: Barrier Options. Survival Probabilities and CDS

Write the price of the option by computing the expected value. We find, if $T < \bar{T}$,

$$\bar{P}(0,T) = P(0,T)\left[\Phi\left(\frac{\ln\left(\frac{V_0}{H(0)}\right) + \tilde{\nu}T}{\sigma_V\sqrt{T}}\right) - \left(\frac{H(0)}{V_0}\right)^{2\tilde{a}}\Phi\left(\frac{\ln\left(\frac{H(0)}{V_0}\right) + \tilde{\nu}T}{\sigma_V\sqrt{T}}\right)\right]$$

where $\tilde{\nu} = r - k - \gamma - \frac{1}{2}\sigma_V^2$ and $\tilde{a} = \frac{\tilde{\nu}}{\sigma_V^2}$. Since $\bar{P}(0,T) = P(0,T)\,\mathbb{Q}\{\tau > T\}$,

$$\mathbb{Q}\{\tau > T\} = \Phi\left(\frac{\ln\left(\frac{V_0}{H(0)}\right) + \tilde{\nu}T}{\sigma_V\sqrt{T}}\right) - \left(\frac{H(0)}{V_0}\right)^{2\tilde{a}}\Phi\left(\frac{\ln\left(\frac{H(0)}{V_0}\right) + \tilde{\nu}T}{\sigma_V\sqrt{T}}\right).$$

Finally we recall that from survivals we can compute CDS prices:

$$\mathrm{CDS}_{0,b}(0, R, \mathrm{L_{GD}}) = \mathrm{accrual} - R\sum_{i=1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)} - \mathrm{L_{GD}}\int_0^{T_b} P(0,t)\,d\boxed{\mathbb{Q}(\tau > t)}$$

Black Cox: Barrier Options. Survival Probabilities and CDS

Thus we have CDS formulas also in Black and Cox.

Black and Cox devoted part of their work to further describing the capital structure of the firm, looking for the best way to express debt.

In fact the zero coupon bond debt assumption is not always satisfying. They alternatively derived a closed form expression for the debt seen as a consol bond, i.e. a bond paying a continuous coupon c for all the life of the firm. The value of the equity can be derived by subtraction: $S(V(t)) = V(t) - D(V(t))$.

Potentially interesting for hybrid equity/credit products with structural models.

But even with its positive features as seen so far, the Black Cox models has one problem: Can one make the model consistent with liquid CDS data by inverting the model formula given market quotes (CDS Calibration)?

Black Cox: CDS Calibration

Can one make the B&C model reproduce liquid CDS data (CDS Calibration)?

$$\left.\begin{array}{c} R_{0,1y}^{\text{MktCDS}} \\ R_{0,2y}^{\text{MktCDS}} \\ \vdots \\ R_{0,10y}^{\text{MktCDS}} \end{array}\right\} \longleftrightarrow \left\{ \begin{array}{c} dV(t) = (r-k)V(t)dt + \sigma_V V(t) dW(t) \\ H(t) = \left\{ \begin{array}{ll} L & t = \bar{T} \\ K e^{-\gamma(\bar{T}-t)} & t < \bar{T} \end{array} \right. \\ \text{model parameters: } \sigma_V, \;\; L, \;\; K, \;\; \gamma \end{array} \right.$$

Typically one has from 5 to 10 CDS market quotes and just 4 parameters in the B&C model to calibrate them. Further, even if we had only 4 CDS quotes, the 4 parameters

$$\sigma_V, \;\; L, \;\; K, \;\; \gamma$$

are not much flexible.

Can we extend the model to make it more flexible and capable of exactly retrieving any number of quoted CDS?

Structural Models: CDS Calibration?

Our strategy: (CDS Calibration?)

$$\left.\begin{array}{c} R_{0,1y}^{\text{MktCDS}} \\ R_{0,2y}^{\text{MktCDS}} \\ \vdots \\ R_{0,10y}^{\text{MktCDS}} \end{array}\right\} \longleftrightarrow \left\{ \begin{array}{c} dV(t) = (r-k)V(t)dt + \boxed{\sigma_V(t)} V(t)dW(t) \\ H(t) = \dots \\ \text{model parameters: } t \mapsto \sigma_V(t), \quad t \mapsto H(t) \end{array}\right.$$

Now we would have infinite parameters (all the values of $\sigma(t)$, for example) to account for 10 CDS quotes.

The problem is: can we insert a time-dependent V dynamics **and** preserve barrier-like analytical formulas for survival probabilities $\mathbb{Q}(\tau > t)$ (and thus CDS etc?)?

Structural Models: Barrier Options. CDS Calibration?

The difficulties in formulating such a model (like our AT1P below) and the reason why nobody tried it before is that in general Barrier option problems are difficult or impossible in presence of **time-dependent** volatilities or **general curved barriers**.

However, some recent work shows that it is possible to find analytical barrier option prices when the barrier has a particular curved shape depending partly on the time dependent volatility (Lo et al. (2003), Rapisarda (2003)).

Our AT1P model builds on these results: indeed, our curved barrier $\widehat{H}(t)$ will depend on $\sigma_V(t)$.

Analytically tractable 1st passage models (Brigo et al 2004-2006)

AT1P model: Let the risk neutral firm V dynamics and the default barrier $\widehat{H}(t)$ be

$$dV(t) = V(t) \boxed{(r(t) - q(t))} dt + V(t) \boxed{\sigma(t)} dW(t)$$

$$\widehat{H}(t) = H \exp\left(-\int_0^t \left(\boxed{q(s) - r(s)} + B \boxed{\sigma(s)^2} \right) ds \right) = \frac{H}{V_0} \mathbb{E}\left[V_t\right] e^{\left(-B \int_0^t \sigma_s^2 ds\right)}$$

and let the default time τ be **the 1st time V hits $\widehat{H}$ from above**, starting from $V_0 > H$. Here $H > 0$ and B are free parameters we may use to shape the barrier.

Then the survival probability is given analytically by

$$\mathbb{Q}\{\tau > T\} = \Phi\left(\frac{\log \frac{V_0}{H} + \frac{2B-1}{2} \int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}} \right) - \left(\frac{H}{V_0}\right)^{2B-1} \Phi\left(\frac{\log \frac{H}{V_0} + \frac{2B-1}{2} \int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}} \right).$$

The AT1P Structural Model: The Barrier

The default barrier H_t varies in time, following company and market conditions:
$B = \ (1+2\beta)/2$,

$$H_t \quad = \quad H \exp\left(-\int_0^t \left(q_s - r_s + B\sigma_s^2 \right) ds \right) \tag{14}$$

$$= \quad \boxed{\frac{H}{V_0} E^{\mathbb{Q}}\left[V_t\right]} \quad \times \quad \underbrace{\exp\left(-B \int_0^t \sigma_s^2 ds \right)}$$

Backbone of the Barrier, proportional to expected asset value

Cutting some slack in high volatility condition

CDS Calibration with the structural model

$\mathbb{Q}\{\tau > T\} =$ formula in $(\sigma(\cdot), H, B)$. Recall:

$$\mathrm{CDS}_{0,b}(0, R, \mathrm{L}_{\mathrm{GD}}) = R \sum_{i=1}^{b} \int_{T_{i-1}}^{T_i} P(0,t)(t - T_{i-1}) d\mathbb{Q}(\tau > t)$$

$$-R \sum_{i=1}^{b} P(0,T_i)\alpha_i \mathbb{Q}(\tau \geq T_i) - \mathrm{L}_{\mathrm{GD}} \int_0^{T_b} P(0,t) d\mathbb{Q}(\tau > t)$$

so that our models have formulas for CDS. **CDS market quotes can then be reverse-engineered to calibrate the model to CDS data.**

The AT1P model is the first structural model having the same calibration power of the intensity models, being able to fit the whole term structure of CDS market quotes.

Further, in some stressed situations it may present a higher calibration power than the intensity models. We see this when discussing the Parmalat crisis case study.

The AT1P Structural Model. Homogeneity and V/H

Consider again the AT1P formula (here and in the following $\beta = B - 1/2$).

$$\mathbb{Q}\{\tau > T\} = \left[\Phi\left(\frac{\log \frac{V_0}{H} + \beta \int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}} \right) - \left(\frac{H}{V_0} \right)^{2\beta} \Phi\left(\frac{\log \frac{H}{V_0} + \beta \int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}} \right) \right].$$

Survival (and default) probabilities (and thus CDS valuation) only depend on the ratio between V and H, and not on V and H separately.

This means that a precise estimation of V is not needed. We can express default probabilities in relative terms, i.e. in terms of H/V.

We will often normalize V to 1 and express H with respect to unity.

The AT1P Structural Model: Calibrating parameters

We have seen: $\mathbb{Q}\{\tau > T\}$ = formula in $(\sigma(\cdot), H, B$ (or β)).

This Formula can be used to fit the model parameters to market data by inverting CDS formulas in correspondence of market quotes ("reverse engineering").

However, the only parameter left that can account for time dependence is the volatility, since $r(t) - q(t)$ is determined by risk neutrality.

If the vol $\sigma_V(t)$ were already specified exogenously as seen before from historical or implied equity volatility (á la Hull Nelken White) we would be left with no parameters to calibrate CDS data.

If we still wish to keep equity data, we may infer the first year vol $\sigma([0, 1y))$ from equity á la Hull, Nelken and White and use H as a 1st fitting parameter for CDS, and then use the remaining later vols $\sigma([1y, 2y)), \sigma([2y, 3y))$ etc as further CDS parameters.

The parameter β can be tuned to help shape safety covenants and reshape the barrier.

The AT1P Structural Model: Calibrating parameters and Finding H

$$\mathbb{Q}\{\tau > T\} = \left[\Phi \left(\frac{\log \frac{V_0}{H} + \beta \int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}} \right) - \left(\frac{H}{V_0} \right)^{2\beta} \Phi \left(\frac{\log \frac{H}{V_0} + \beta \int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}} \right) \right].$$

Here is a possible method to find H:

Take and exogenous candidate $\sigma([0, 1y))$ from equity data, and

the 1y default probability as stripped from CDS (for example with an *intensity model*);

Use the constant coefficients and barrier formula for $\mathbb{Q}(\tau > T)$ with an unknown H and $\sigma = \sigma([0, 1y))$, finding H that matches the given $1y$ prob from CDS.

Having H, start the calibration of all CDS quotes based on σ.

We refer to this as to the "***credit spread* method**" for H.

The AT1P Structural Model: Calibrating Parameters

This procedure is justified: we are not interested in estimating the real process of the firm value but only in **reproducing risk neutral default probabilities with a model that makes sense also economically**.

We are interested in the economic interpretation and not in sharply estimating the capital structure of the firm.

We appreciate the structural model interpretation as a tool for assessing the realism of the outputs of calibrations, and as an instrument to check economic consequences and possible diagnostics.

The AT1P Structural Model: Calibrating Parameters

In the calibration, we can also arbitrarily choose the value H according to direct considerations instead of using the earlier method based on CDS/equity data.

Also β can be tuned to help shape safety covenants, leaving all the unknown information in the calibration of the volatility.

If we do so, we find exactly one volatility parameter for each CDS maturity, including the first one. **This creates a one-to-one correspondence between CDS quotes and value of the firm volatilities.** We used this approach in the Parmalat case study presented in the following, where we fixed $H = 0.4$ based on recovery considerations.

$H = 0.4$: If default, the CDS pays $1 - \text{Rec}$, covering the experienced Loss.

The Loss in the firm value at the beginning, in case of instantaneous default, would be $V_0 - H = 1 - H$.

So we set $H = \text{Rec}$ and calibrate CDS only through σ.

The AT1P Structural Model: Final Debt

Finally, if we have a preferred terminal debt maturity $\bar{T}$, we can still have an indication of the final debt at maturity, which we call L.

It makes sense then to impose that our barrier $\widehat{H}$ be always below $P(t, \bar{T})L$ in $t < \bar{T}$. This condition amounts to assuming that we are cutting some slack to the firm by allowing it to go somehow below the debt present value before forcing it to early bankruptcy. How much below can be decided by means of H and B. We have

$$L > H \exp\left[-\int_0^t \left(q(s) + B\sigma(s)^2\right) ds + \int_0^{\bar{T}} r(s) ds\right]$$

In our use of the model typically $\bar{T} > 10y$, since this is the largest CDS maturity. A sufficient condition for the above inequality is, if the round brackets term is positive, is $H < LP(0, \bar{T})$

A Case Study with AT1P: Parmalat Default Story

After a period of uncertainty on Parmalat's real financial situation, due to deceptive accounting, the real depth of the financial crisis came to light at the end of 2003 and rapidly led to bankruptcy.

September 12, 2003: Parmalat drops plan for a EUR 300 million debt sale.

November 14, 2003: the chief financial officer resigns after questions have been raised on Parmalat financial transactions.

December 9, 2003: Parmalat misses a EUR 150 million bond payment, while the management claims this is due to a customer not paying its bills.

December 19, 2003: a claimed USD 3.9 billion liquidity is revealed not to exist.

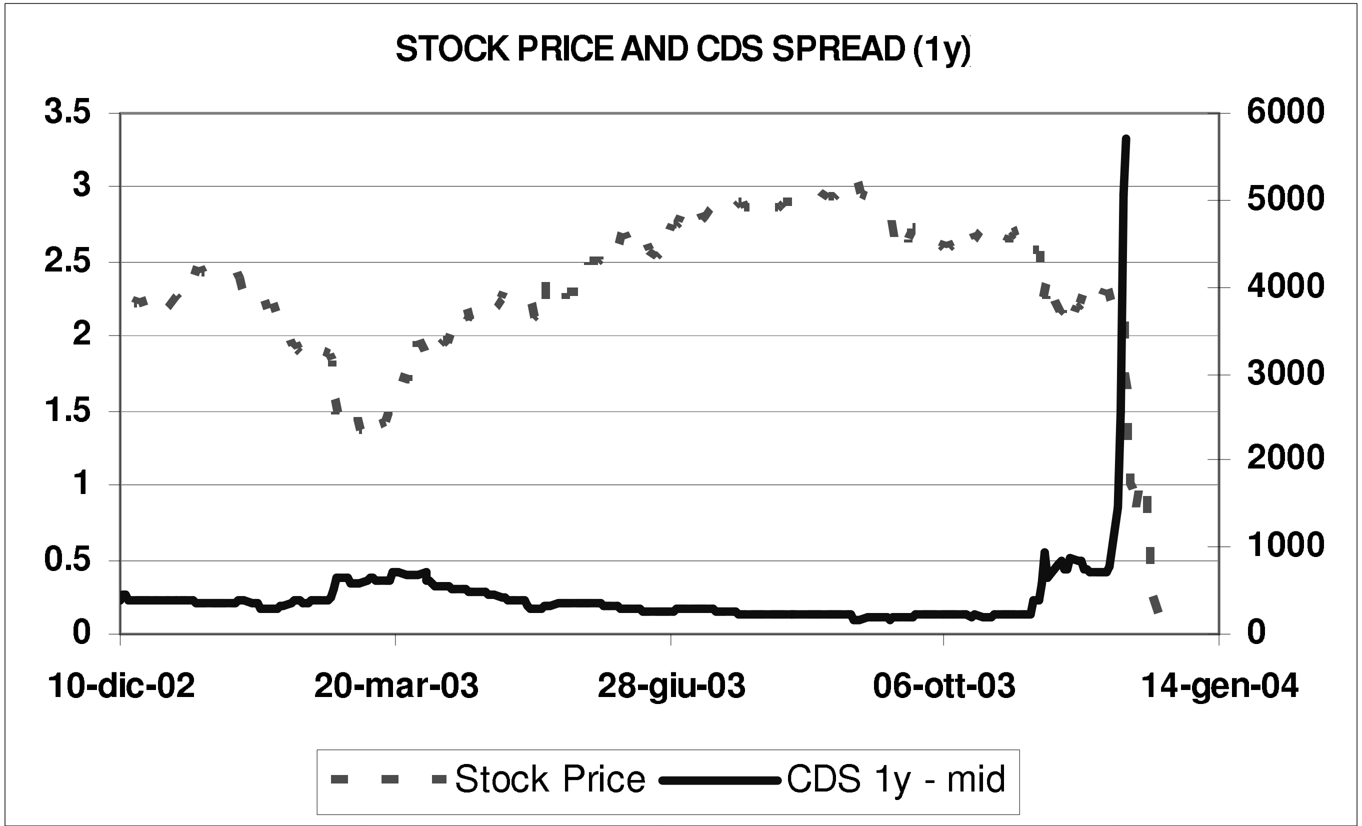
December 24, 2003: Parmalat goes into administration.

Parmalat CDS data

We consider data from three different days in 2003:
1. **September 10**, just before the beginning of the final Parmalat default story recalled above.
2. **November 28**, after the story of the Parmalat crisis began to unfold but before the pitch of the crisis.
3. **December 10**, when the fraud was not clear yet but the company was openly suspected to be on the verge of bankruptcy.

CDS Maturity T_b	1y	3y	5y	7y	10y
September 10: Premium rate $R_{0,b}$	192.5	215	225	235	235
November 28: Premium rate $R_{0,b}$	725	630	570	570	570
December 8: Premium rate $R_{0,b}$	1450	1200	940	850	850
December 10: Premium rate $R_{0,b}$	5050	2100	1500	1250	1100

Parmalat: CDS quotes and stock prices behavior



Behavior of the stock price (in euros, left scale) and of the 1y-CDS spread (in bps., right scale).

A Case Study: Structural AT1P vs Deterministic Intensity Parmalat CDS Calibration

The recovery is set to 40% ($\text{L}_{\text{GD}} = 0.6$) at all dates except in full crisis (December 8 and 10), where we set it to 25% and 15% respectively. The payout ratio is set to 0.

$r(t)$ comes from the zero curve. In all cases $H = 0.4$ and $\beta = 0.2$ (or $B = 0.7$).

September 10th, 2003.

The volatilities from the calibration are large (28%-44%). The reason is due to the fact that we chose a low barrier level, thus leading to high volatilities. Also the lognormal distribution does not display fat tails that may spread trajectories without resorting to huge volatility values. We will see later on that a scenario model, implying fatter (mixture) tails, leads to more reasonable volatilities.

A Calibration Case Study: Parmalat, September 10th, 2003

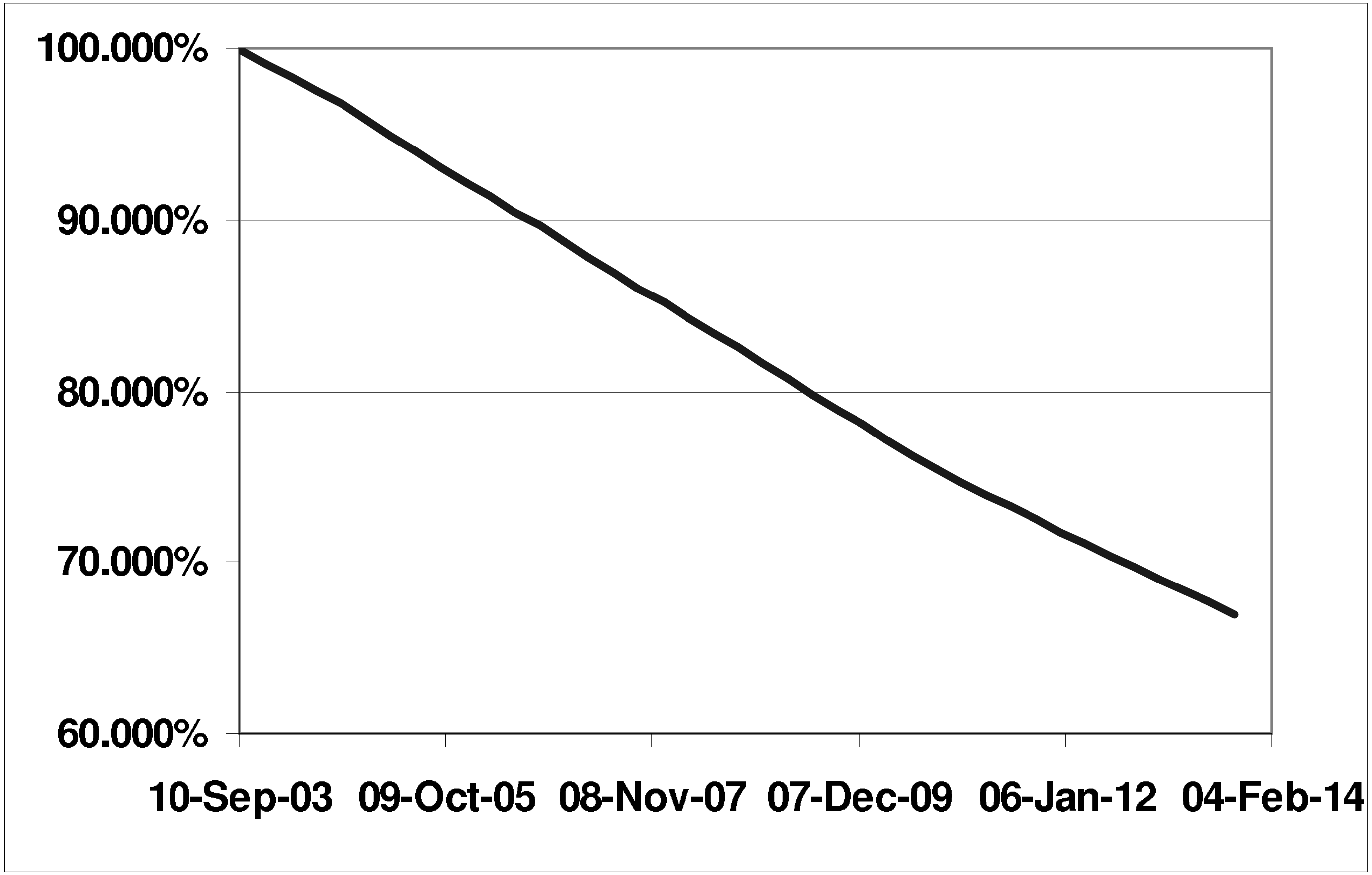
CDS maturity T_b	Spread $R_{0,b}(0)$ (bps.)
1y	192.5
3y	215
5y	225
7y	235
10y	235

CDS maturity T_b	Intensity	Survival
Today	3.199%	100.000%
1y	3.199%	96.714%
3y	3.780%	89.578%
5y	4.033%	82.516%
7y	4.458%	75.402%
10y	3.891%	66.978%

CDS maturity T_b	Volatility	Survival
Today	43.669%	100.000%
1y	43.669%	96.675%
3y	28.102%	89.530%
5y	29.660%	82.483%
7y	34.225%	75.392%
10y	36.792%	67.032%

A Calibration Case Study: Parmalat, September 10th, 2003

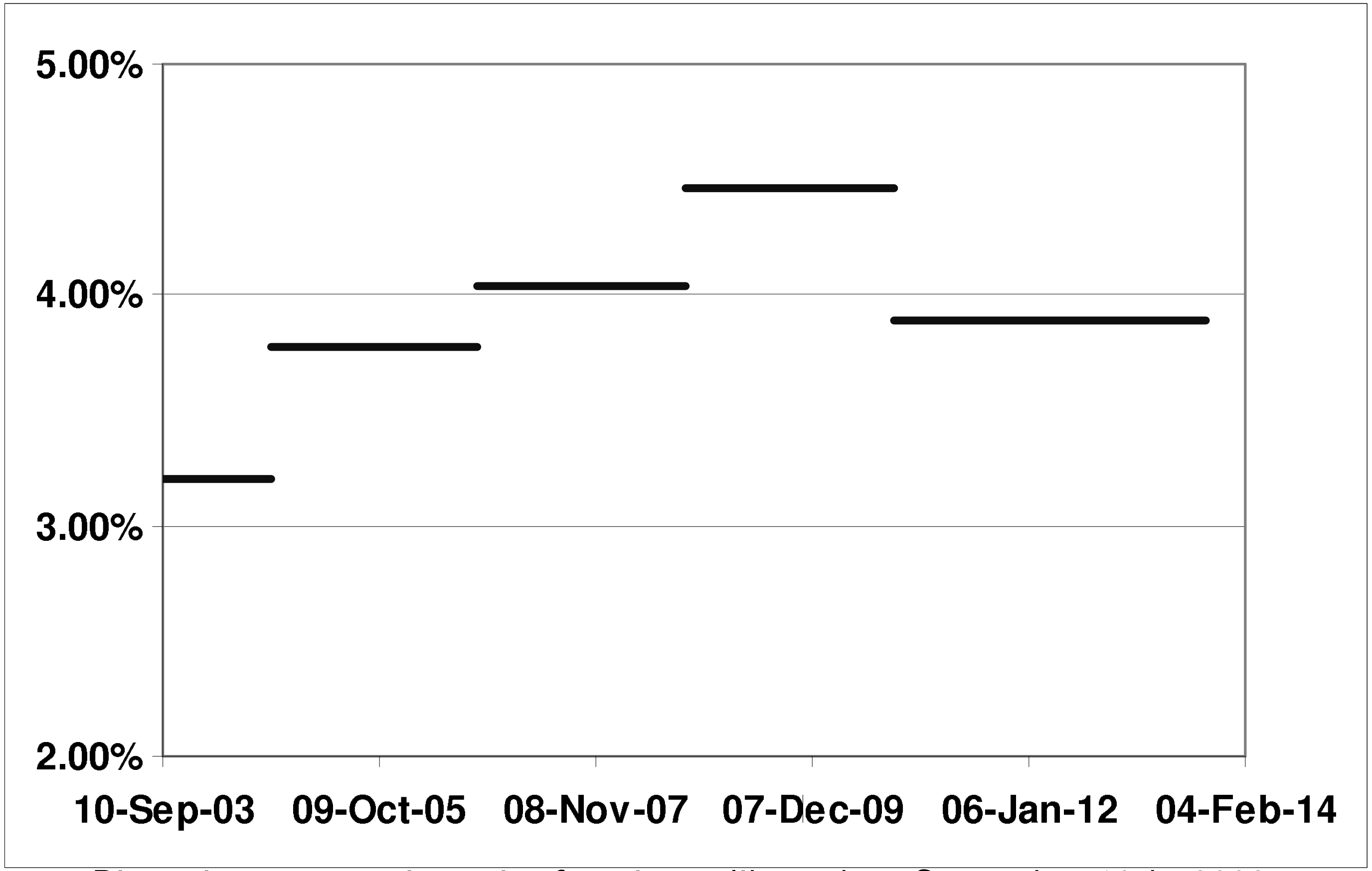
Survival Probability



Survival probability computed with the (piecewise constant) intensity model on September 10th, 2003.

A Calibration Case Study: Parmalat, September 10th, 2003

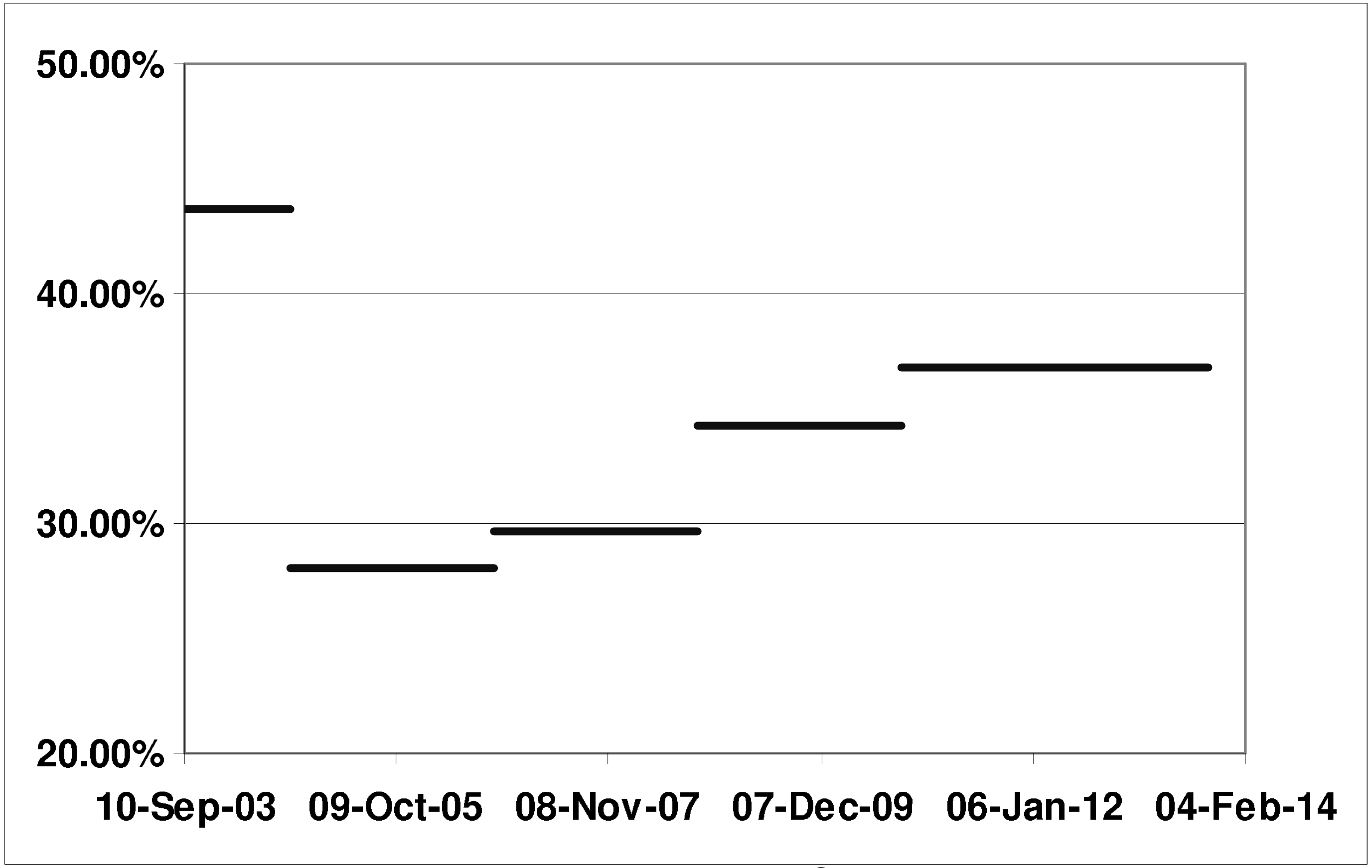
Calibrated piecewise constant intensity



Piecewise constant intensity function calibrated on September 10th, 2003.

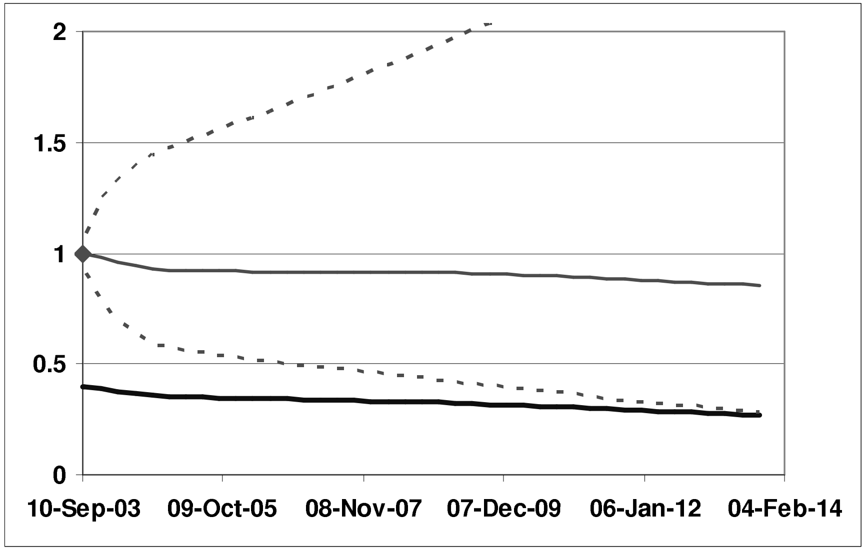
A Calibration Case Study: Parmalat, September 10th, 2003

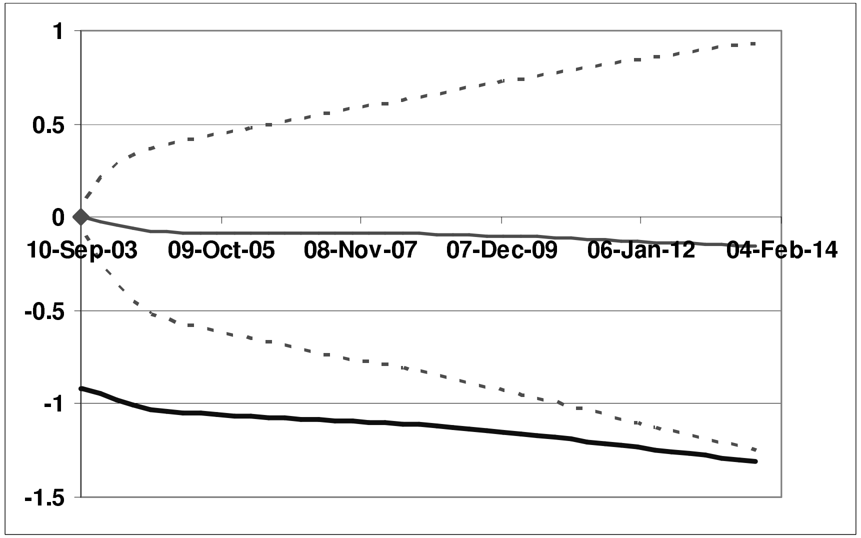
Calibrated volatility



Volatility term structure calibrated on September 10th, 2003.

A Calibration Case Study: Parmalat, September 10th, 2003





Default barrier on September 10th, 2003, plotted together with the median value $t \mapsto \exp(\mathbb{E}[\ln(V(t))])$ and a confidence region $t \mapsto \exp(\mathbb{E}[\ln(V(t))] \mp \text{Std}[\ln(V(t))])$.

On the left we plot the absolute value, on the right the logarithmic values.

Second date: November 28th, 2003.

The credit quality has deteriorated. Calibrated intensity is higher in the intensity model. Calibrated volatilities get larger in the AT1P model.

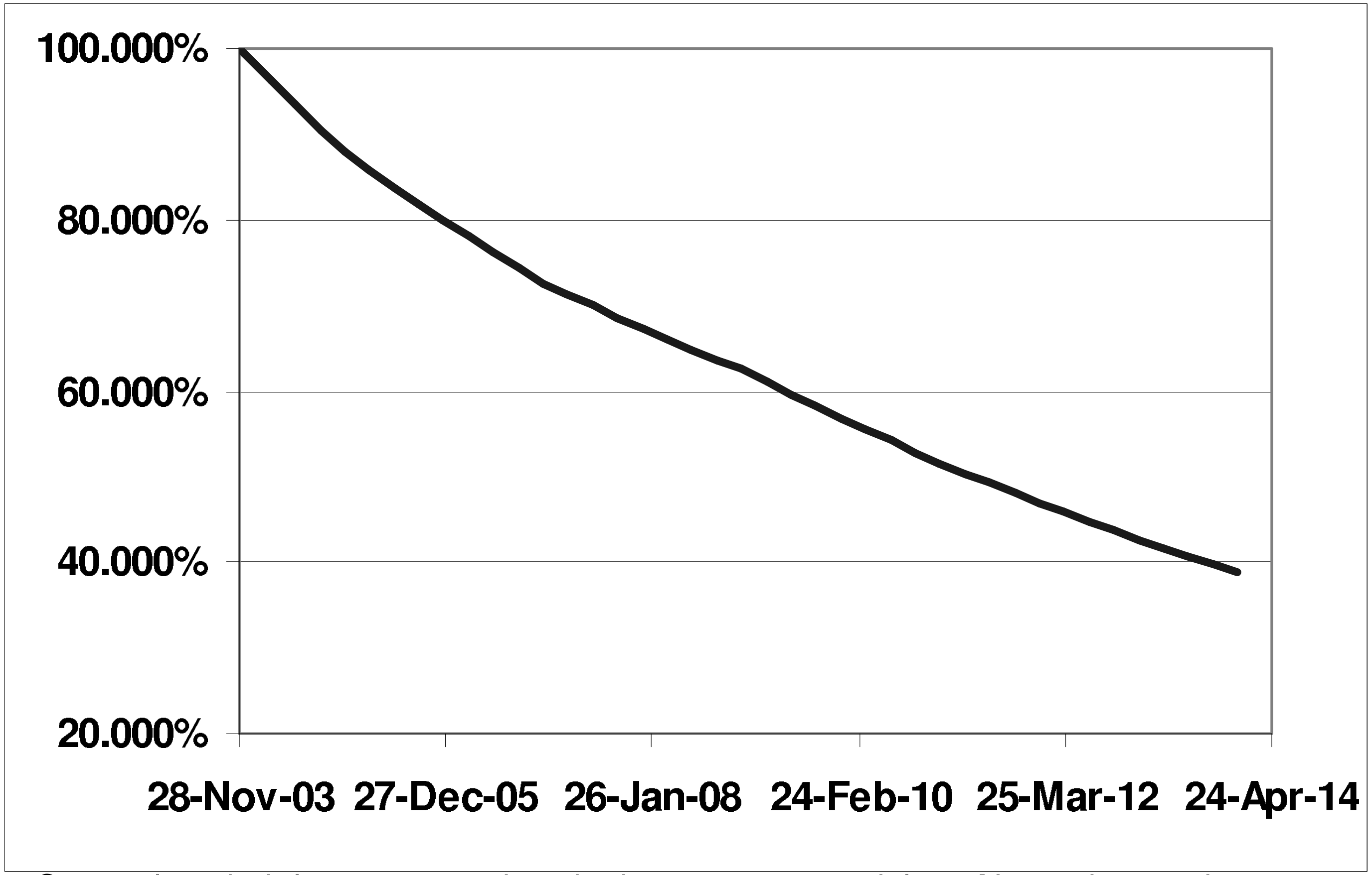
CDS maturity T_b	Spread $R_{0,b}(0)$ (bps.)
1y	725
3y	630
5y	570
7y	570
10y	570

CDS maturity T_b	Intensity	Survival
Today	12.047%	100.000%
1y	12.047%	87.824%
3y	9.426%	72.545%
5y	7.331%	62.486%
7y	9.441%	51.626%
10y	9.437%	38.734%

CDS maturity T_b	Volatility	Survival
Today	61.202%	100.000%
1y	61.202%	87.637%
3y	49.177%	72.569%
5y	54.727%	62.611%
7y	82.487%	51.891%
10y	136.517%	39.470%

Second date: November 28th, 2003.

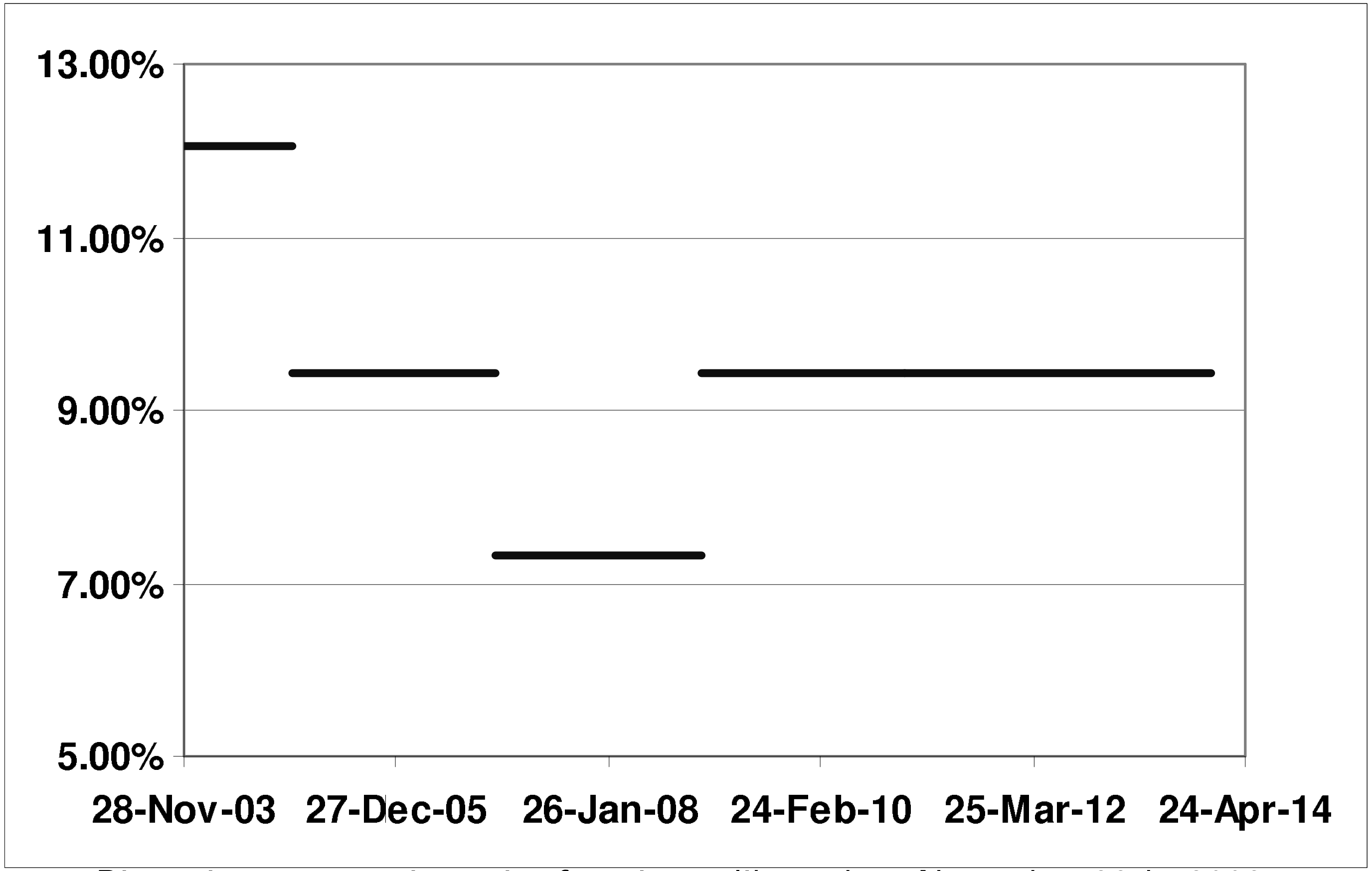
Survival probability



Survival probability computed with the intensity model on November 28th, 2003.

Second date: November 28th, 2003.

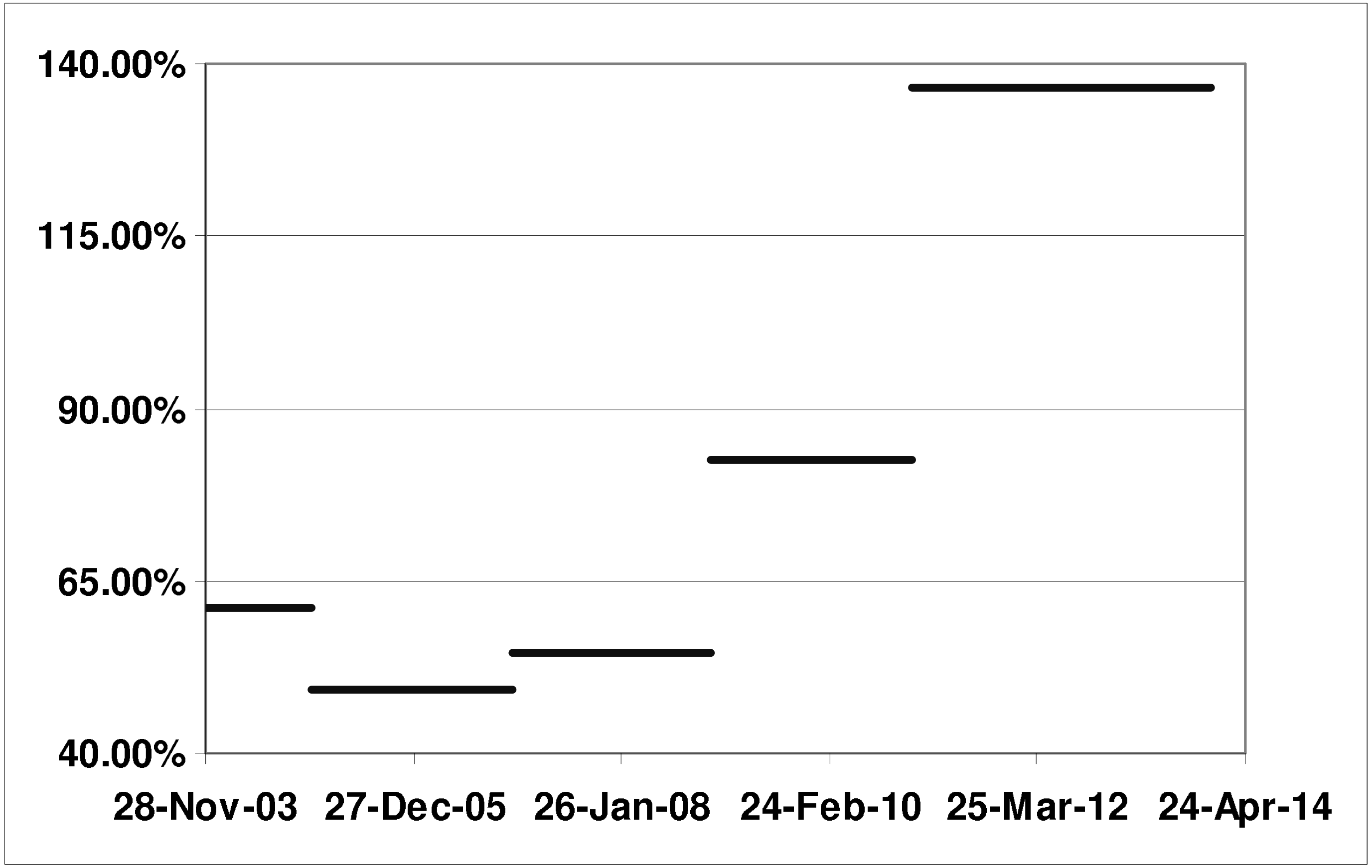
Calibrated piecewise constant intensity



Piecewise constant intensity function calibrated on November 28th, 2003.

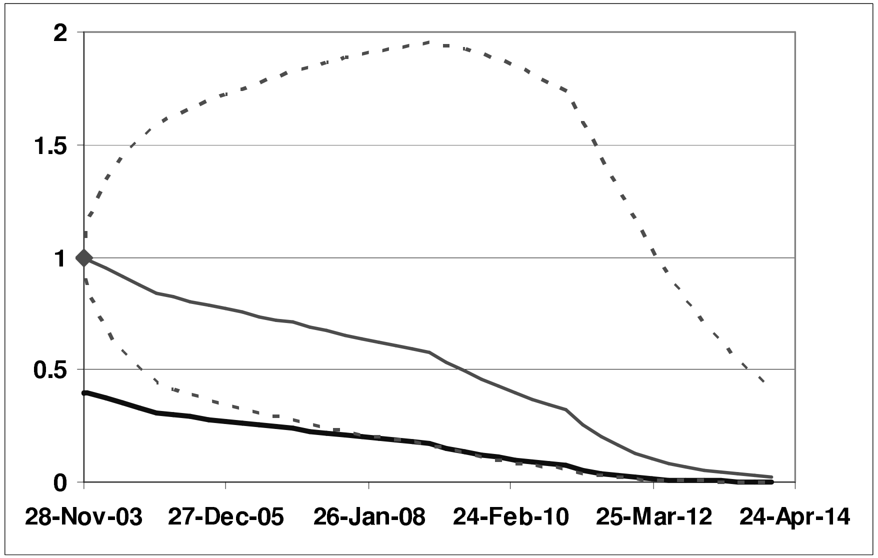
Second date: November 28th, 2003.

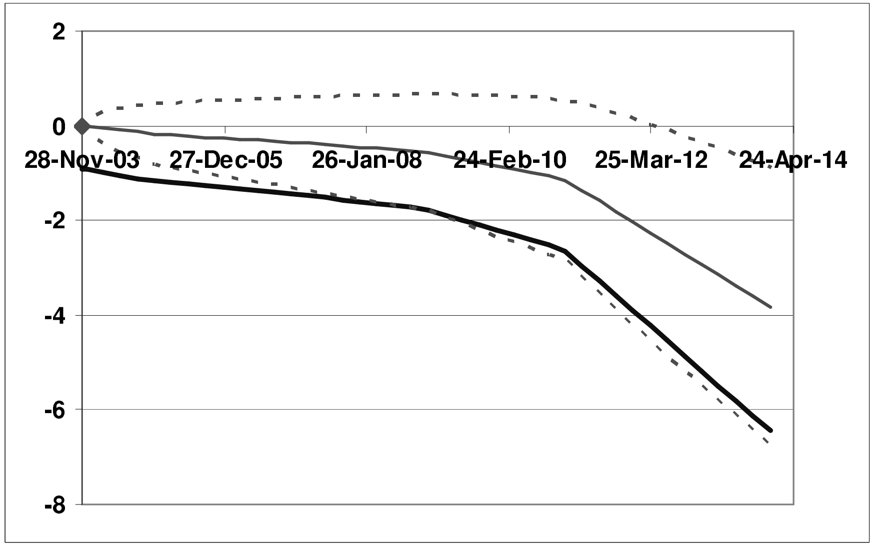
Calibrated volatility



Volatility term structure calibrated on November 28th, 2003.

Second date: November 28th, 2003.





Default barrier on November 28th, 2003, plotted together with the median value $t \mapsto \exp(\mathbb{E}[\ln(V(t))])$ and a confidence region $t \mapsto \exp(\mathbb{E}[\ln(V(t))] \mp \mathrm{Std}[\ln(V(t))])$.

On the left we plot the absolute value, on the right the logarithmic values.

Third date: December 8th, 2003.

Credit quality has deteriorated further.
Larger calibrated intensities and larger AT1P calibrated volatilities.

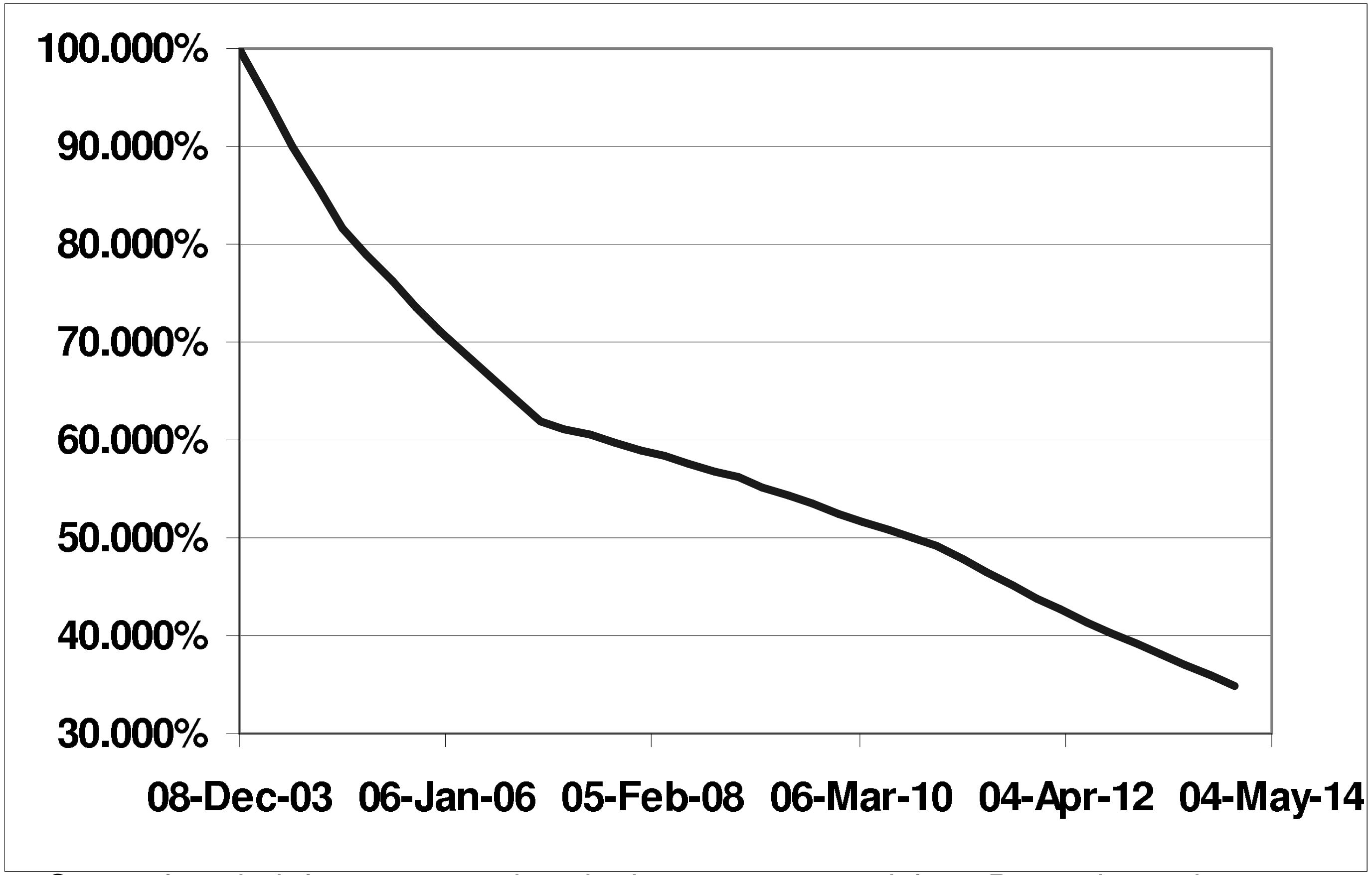
CDS maturity T_b	Spread $R_{0,b}(0)$ (bps.)
1y	1450
3y	1200
5y	940
7y	850
10y	850

CDS maturity T_b	Intensity	Survival
Today	19.272%	100.000%
1y	19.272%	81.680%
3y	13.650%	61.931%
5y	4.834%	56.126%
7y	6.500%	49.213%
10y	11.256%	34.934%

CDS maturity T_b	Volatility	Survival
Today	73.830%	100.000%
1y	73.830%	81.324%
3y	69.189%	62.145%
5y	55.191%	56.321%
7y	78.971%	49.447%
10y	186.312%	35.961%

Third date: December 8th, 2003.

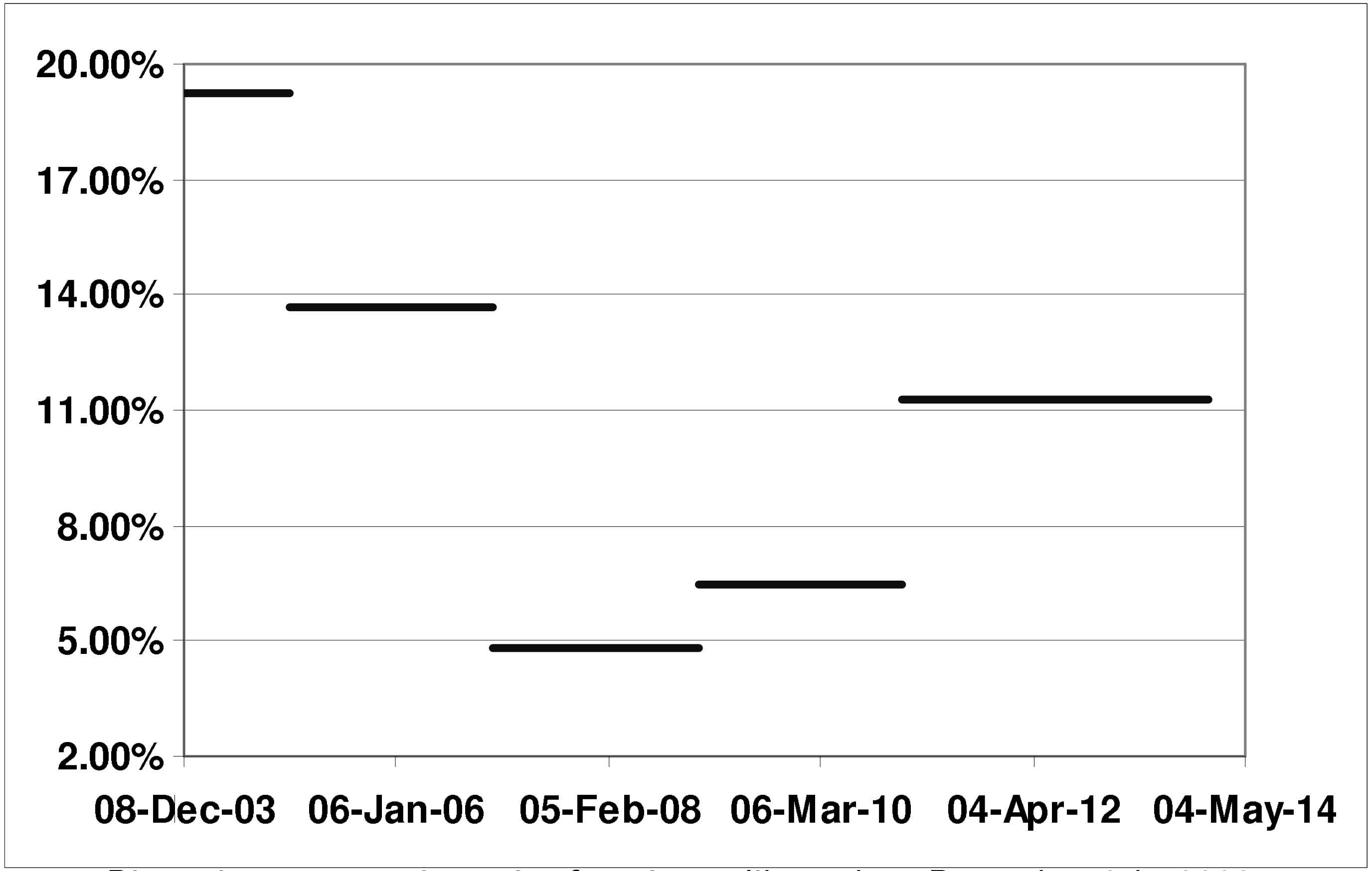
Survival probability



Survival probability computed with the intensity model on December 8th, 2003.

Third date: December 8th, 2003.

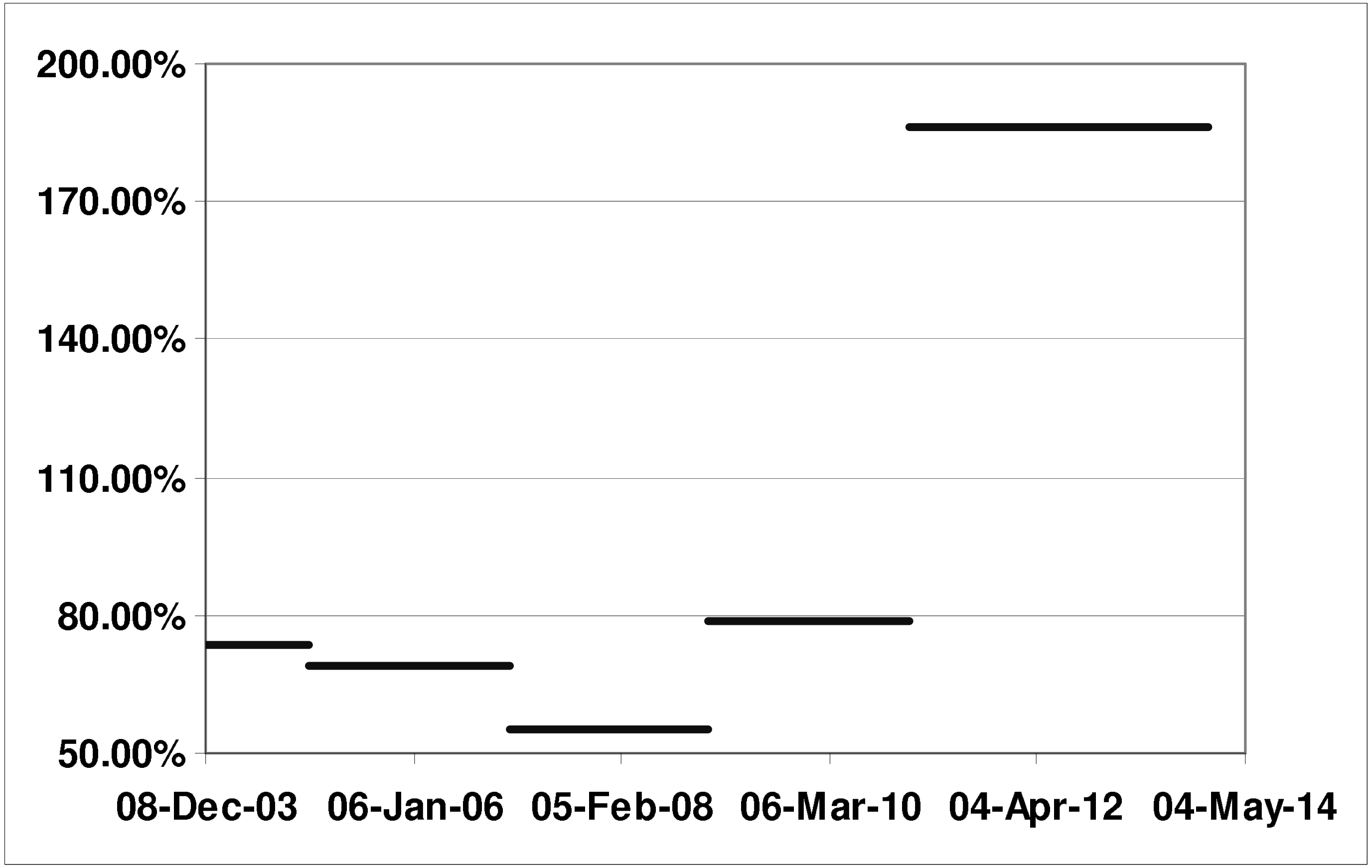
Calibrated piecewise constant intensity



Piecewise constant intensity function calibrated on December 8th, 2003.

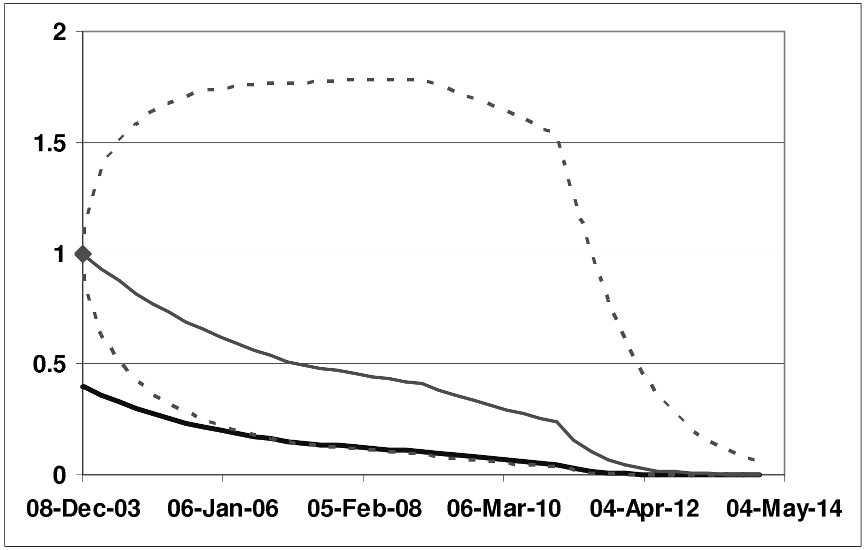
Third date: December 8th, 2003.

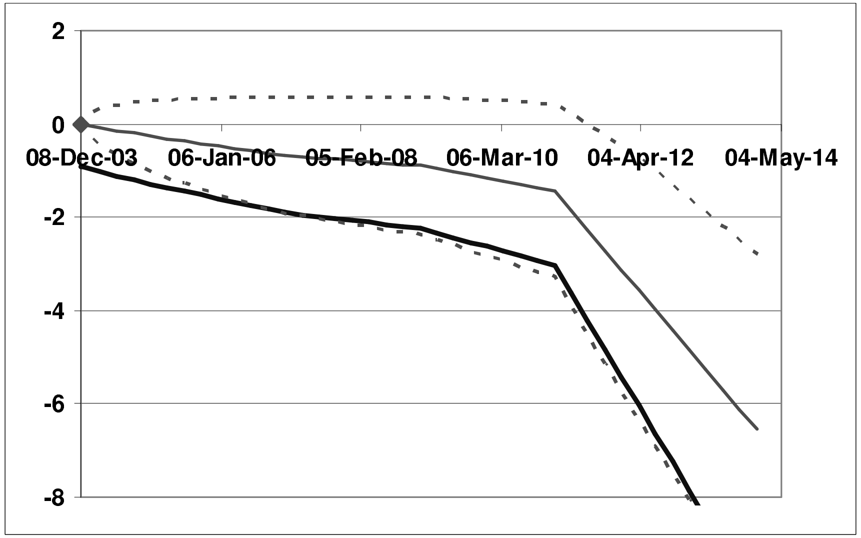
Calibrated volatility



Volatility term structure calibrated on December 8th, 2003.

Third date: December 8th, 2003.





Default barrier on December 8th, 2003, plotted together with the median value $t \mapsto \exp(\mathbb{E}[\ln(V(t))])$ and a confidence region $t \mapsto \exp(\mathbb{E}[\ln(V(t))] \mp \text{Std}[\ln(V(t))])$.

On the left we plot the absolute value, on the right the logarithmic values.

Fourth date: December 10th, 2003.

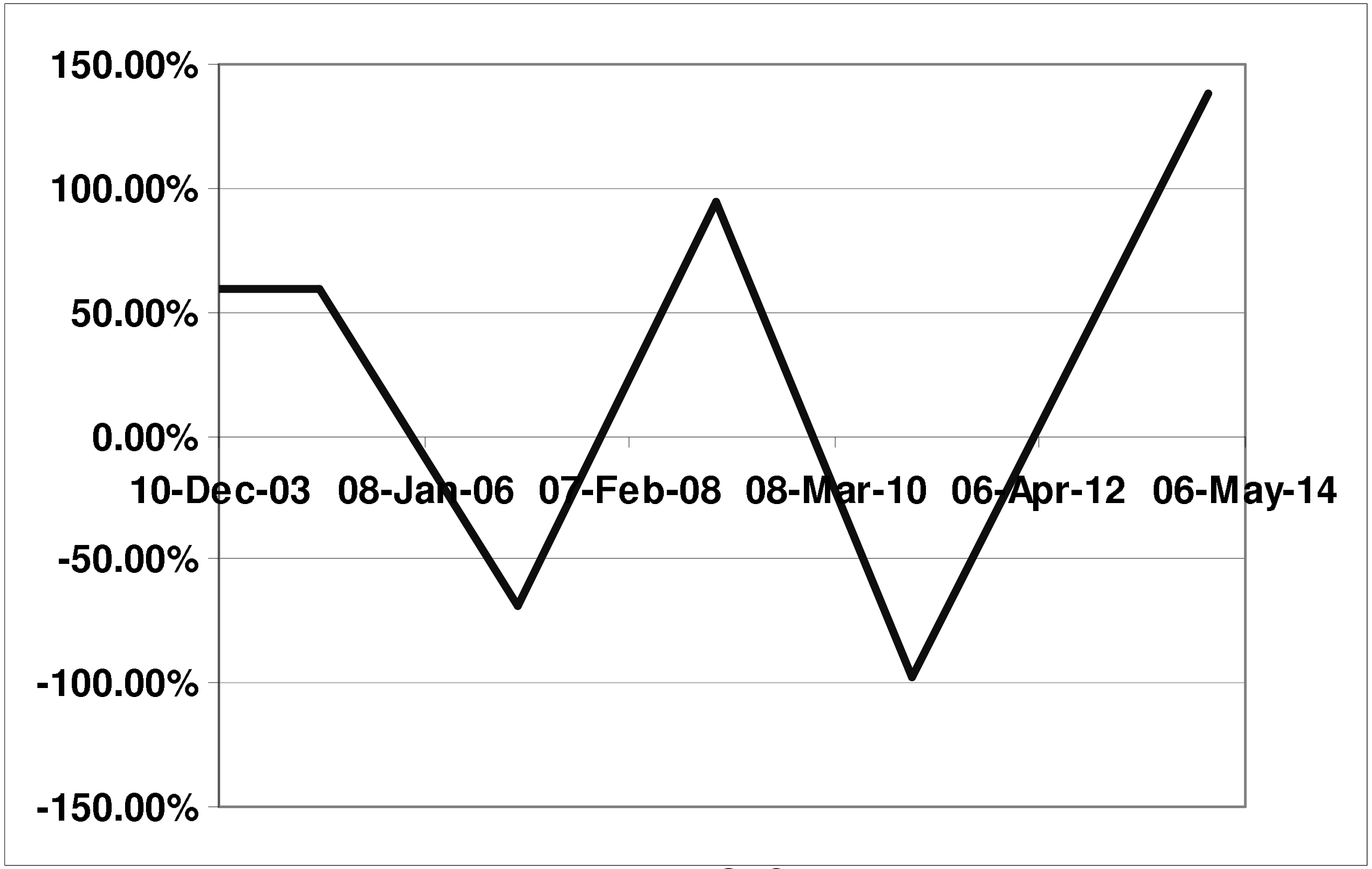
Here we have practically reached the crisis.
The intensity model calibration possibly returns negative intensities. The structural model is still capable of calibration. We see that in the calibrated structural model volatilities are very large and erratic.

CDS maturity T_b	Spread $R_{0,b}(0)$ (bps.)
1y	5050
3y	2100
5y	1500
7y	1250
10y	1100

CDS maturity T_b	Volatility	Survival
Today	146.967%	100.000%
1y	146.967%	55.642%
3y	16.956%	55.233%
5y	61.284%	50.926%
7y	75.348%	46.720%
10y	123.394%	39.177%

Fourth date: December 10th, 2003.

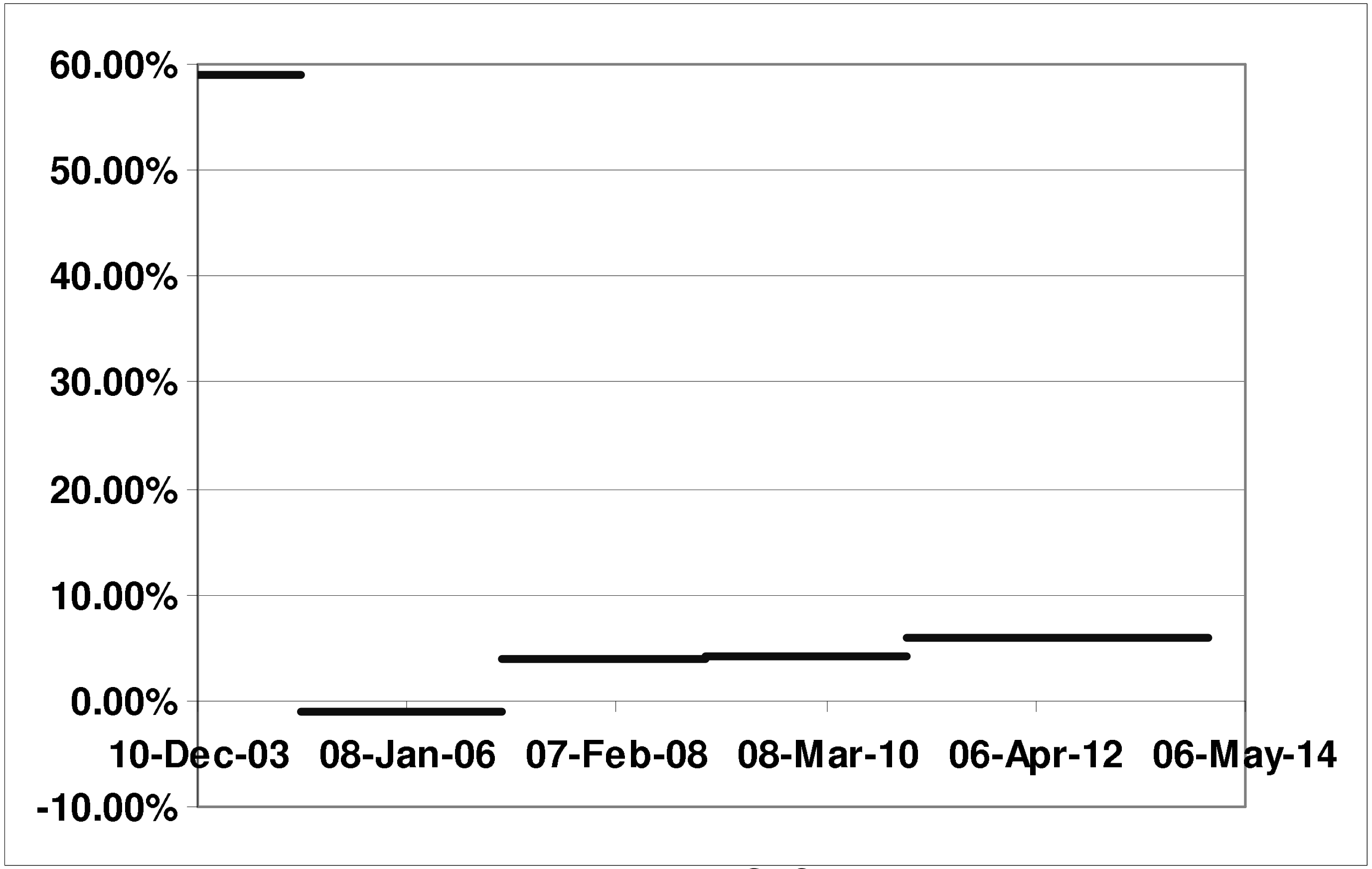
Calibrated piecewise linear intensity



Piecewise linear intensity γ calibrated on CDS quotes on December 10th, 2003.

Fourth date: December 10th, 2003.

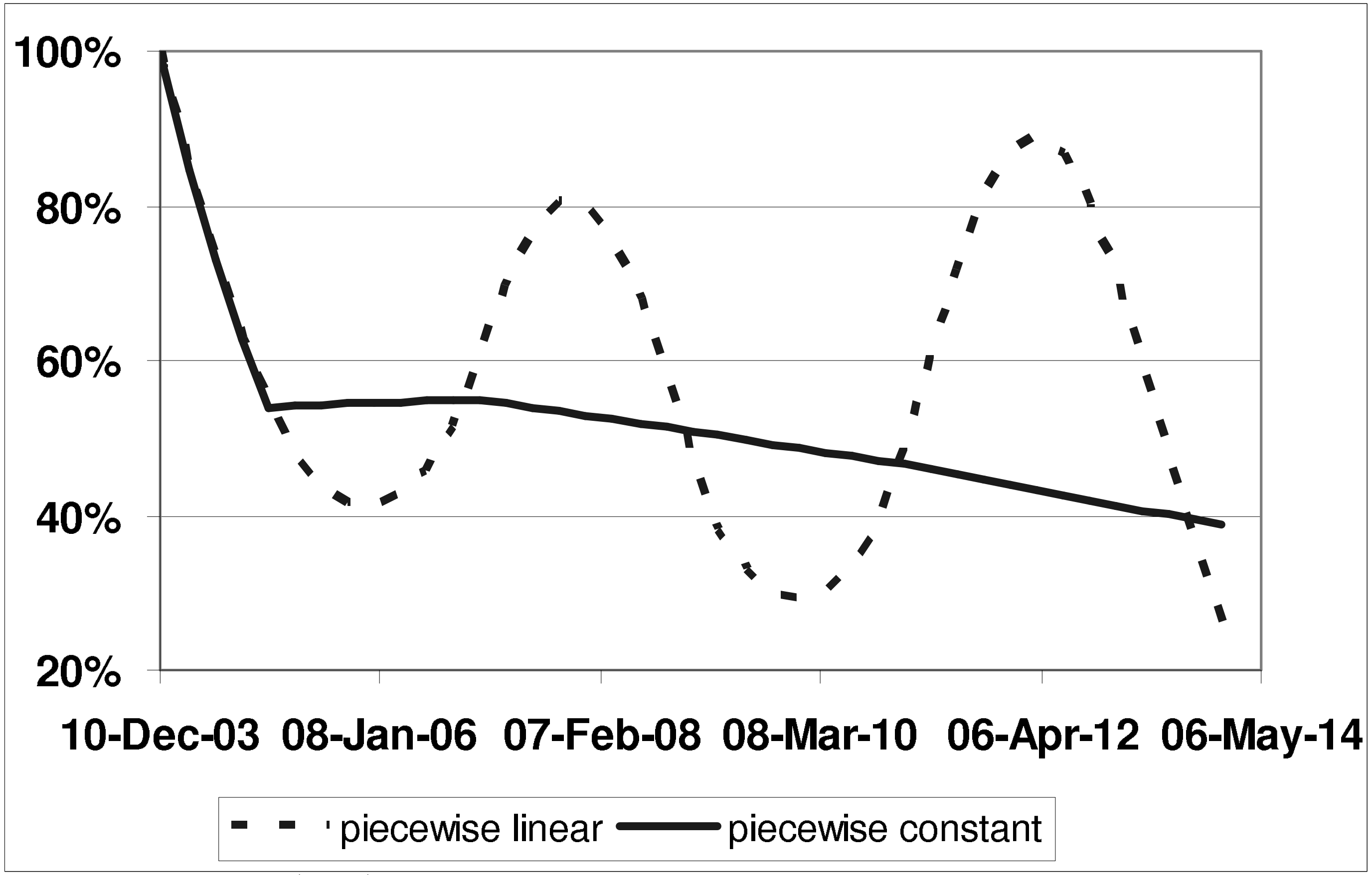
Calibrated piecewise constant intensity



Piecewise constant intensity γ calibrated on CDS quotes on December 10th, 2003.

Fourth date: December 10th, 2003.

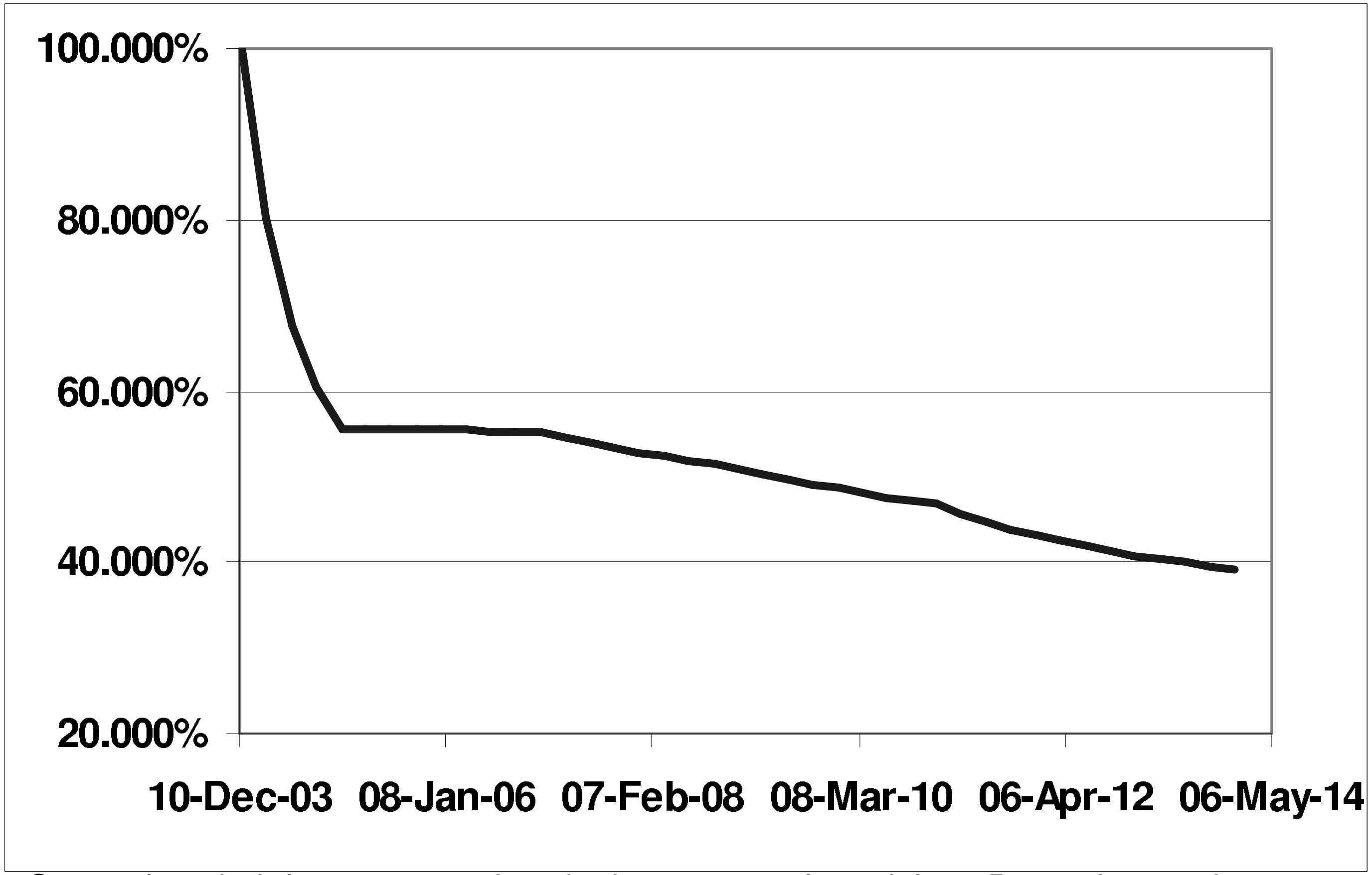
Intensity model: comparison of survival probabilities



Survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on December 10th, 2003.

Fourth date: December 10th, 2003.

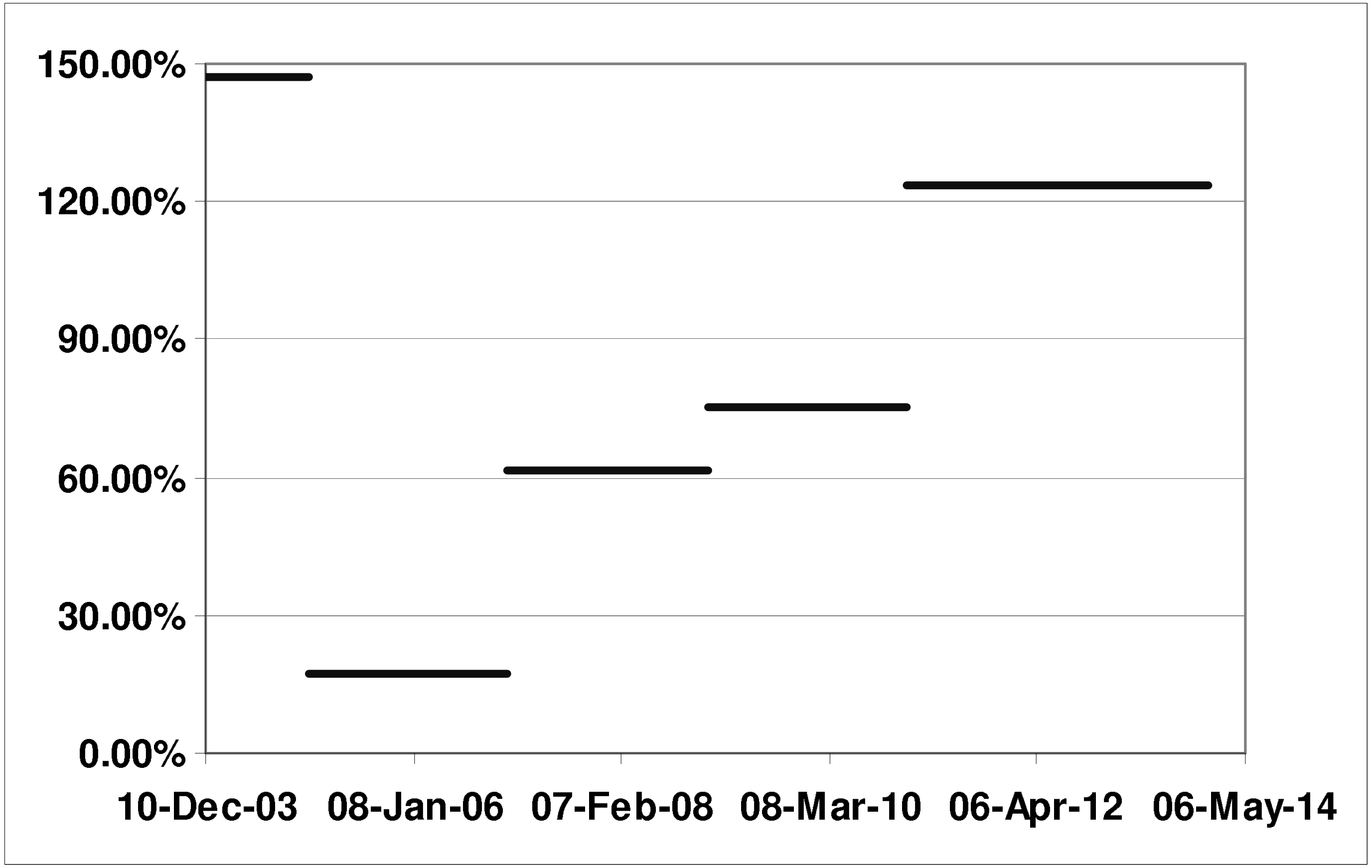
Structural model: survival probability



Survival probability computed with the structural model on December 10th, 2003.

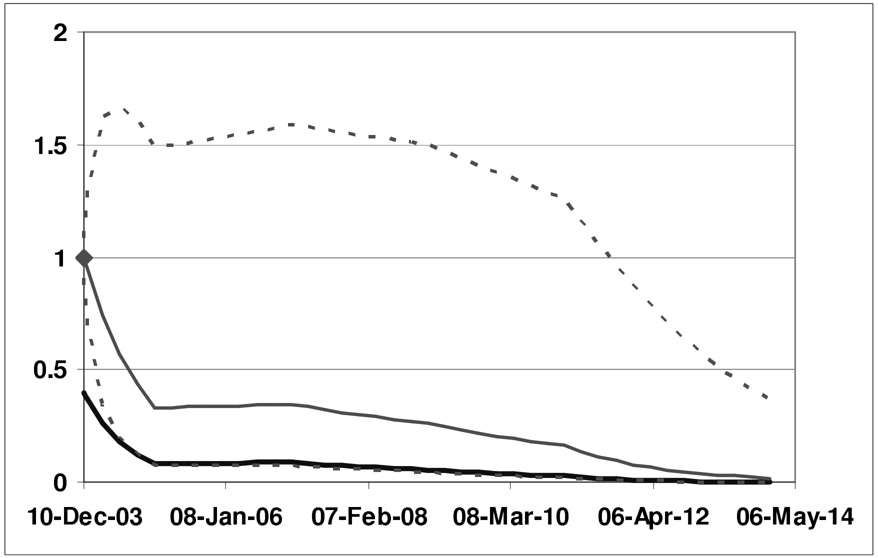
Fourth date: December 10th, 2003.

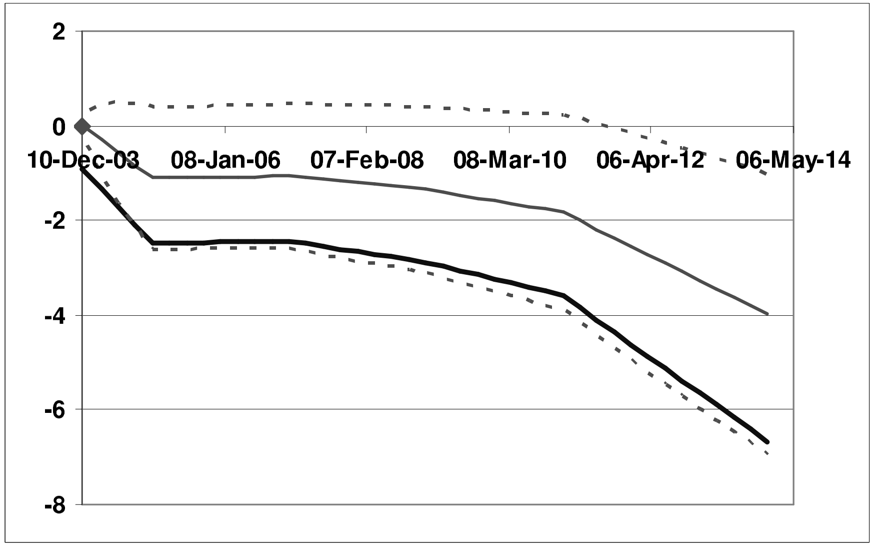
Calibrated volatilities



Volatility term structure calibrated on December 10th, 2003.

Fourth date: December 10th, 2003.





Default barrier on December 10th, 2003, plotted together with the median value $t \mapsto \exp(\mathbb{E}[\ln(V(t))])$ and a confidence region $t \mapsto \exp(\mathbb{E}[\ln(V(t))] \mp \mathsf{Std}[\ln(V(t))])$.

On the left we plot the absolute value, on the right the logarithmic values.

Extensions: Random Barrier and Random Volatility

In some cases it can be interesting to keep the volatility of the process V as an exogenous input coming from the equity and debt worlds (for example it could be related to an historical or implied volatility, á la Hull Nelken White above).

Or we might retain a time-varying volatility to be used only partly as a fitting parameter.

Or we might wish to remove time-varying volatility to avoid an all-fitting approach that is dangerous for robustness.

In all cases we would need to introduce other calibration parameters into the model to compensate for the loss of degrees of freedom under any of these choices.

Where can these calibration parameters be introduced in the model?

Extensions: Random Barrier and Random Volatility

One possibility: introduce a random default barrier.

Intuition: balance sheet information is not certain, possibly because the company is hiding information.

Same AT1P model as before, **but now the default barrier level H is replaced by a random variable H assuming different scenarios with given risk neutral probabilities**.

At a second stage, we may introduce volatility scenarios as well.

By imposing time-constant volatility scenarios, in each single scenario we lose flexibility with respect to the general AT1P, but regain flexibility thanks to the multiple scenarios on otherwise too simple time-constant volatilities.

Even so, we will see that the time-constant scenario model results in a less flexible structure than the old deterministic time-varying AT1P model as far as CDS calibration is concerned (see Brigo and Tarenghi (2005) for a detailed treatment of the extension of the AT1P model to random barrier and volatility).

Extensions: Random Barrier/Volatility. SVBAT1P model

SVBAT1P: Let the risk neutral firm V dynamics and the default barrier $\widehat{H}(t)$ be

$$dV(t) = V(t)\,(r(t) - q(t))\,dt + V(t)\,\sigma(t)\,dW(t)$$

$$\widehat{H}(t) = H \exp\left(-\int_0^t \left(q(s) - r(s) + B\sigma(s)^2\right) ds\right) = \frac{H}{V_0}\mathbb{E}\left[V_t|\sigma, H\right]\, e^{\left(-B\int_0^t \sigma_s^2 ds\right)}$$

and let the default time τ be **the 1st time V hits $\widehat{H}$ from above**, starting from $V_0 > H$. The safety barrier parameter H and the firm value volatility function $t \mapsto \sigma(t)$ may assume scenarios $(H_1, \sigma^1), \ldots, (H_{N-1}, \sigma^{N-1}), (H_N, \sigma^N)$ with $\mathbb{Q}$ probability $p_1, \ldots, p_{N-1}, p_N$. H and σ are assumed to be independent of W.

Then the survival probability is given analytically by $\mathbb{Q}\{\tau > T\} =$

$$= \sum_{j=1}^N p_j \left[\Phi\left(\frac{\log\frac{V_0}{H_j} + \frac{2B-1}{2}\int_0^T \sigma_j(s)^2 ds}{\sqrt{\int_0^T \sigma_j(s)^2 ds}}\right) - \left(\frac{H_j}{V_0}\right)^{2B-1} \Phi\left(\frac{\log\frac{H_j}{V_0} + \frac{2B-1}{2}\int_0^T \sigma_j(s)^2 ds}{\sqrt{\int_0^T \sigma_j(s)^2 ds}}\right)\right].$$

CDS Calibration with the structural model

$\mathbb{Q}\{\tau > T\}$ = formula in $(\sigma_j, H_j, p_j)_{j=1,\ldots,N}, B$. Recall:

$$\mathrm{CDS}_{0,b}(0, R, \mathrm{L}_{\mathrm{GD}}) = R \sum_{i=1}^{b} \int_{T_{i-1}}^{T_i} P(0,t)(t - T_{i-1}) d\mathbb{Q}(\tau > t)$$

$$-R \sum_{i=1}^{b} P(0,T_i)\alpha_i \mathbb{Q}(\tau \geq T_i) - \mathrm{L}_{\mathrm{GD}} \int_0^{T_b} P(0,t) d\mathbb{Q}(\tau > t)$$

so that our models have formulas for CDS. **CDS market quotes can then be reverse-engineered to calibrate the model to CDS data.** We can derive the values of $(\sigma_j, H_j, p_j)_{j=1,\ldots,N}, B$ consistent with market CDS quotes.

Examples follow below.

Extensions: Random Barrier/Volatility. SVBAT1P model

This definition has very general consequences. Indeed, if we are to price a payoff Π based on V, by iterated expectation we have

$$\mathbb{E}[\Pi] = \mathbb{E}\{\mathbb{E}[\Pi|H,\sigma]\} = \sum_{i=1}^{N} p_i \mathbb{E}[\Pi|H = H_i, \sigma = \sigma^i]$$

Now, thanks to independence, the term $\mathbb{E}[\Pi|H = H_i, \sigma = \sigma^i]$ is simply the price of the payoff Π under the model with deterministic barrier and volatility seen earlier in the paper, when the barrier is set to H_i and the volatility to σ^i. This means that, in particular, for CDS payoffs we obtain

$$\mathsf{CDS}_{a,b}(t, R, \mathsf{L}\textsc{gd}) = \sum_{i=1}^{N} \mathsf{CDS}_{a,b}(t, R, \mathsf{L}\textsc{gd}; H_i, \sigma^i) \cdot p_i \qquad (15)$$

where $\mathsf{CDS}_{a,b}(t, R, \mathsf{L}\textsc{gd}; H_i, \sigma^i)$ is the CDS price computed according to the AT1P survival probability formula when H is set to H_i and σ to σ^i.

Extensions: Random Barrier/Volatility. SVBAT1P model

Let us consider now a set of natural maturities for CDS quotes. This is to say that we assume $T_a = 0$ and T_b ranging a set of standard maturities, $T_b = 1y, 3y, 5y, 7y, 10y$. Let us set

$$\text{CDS}_k^i := \text{CDS}_{0,k}(0, R, \text{L}_{\text{GD}}; H_i, \sigma_i),$$

i.e. the CDS with first reset in 0, final maturity T_k and valued under the deterministic barrier model with the barrier set to H_i and the deterministic volatility set to σ^i.

Now assume we aim at calibrating the scenario barrier/volatility model to a term structure of CDS data.

Our first attempt is only with a scenario barrier model SBAT1P, in that we take a **common time-constant volatility** $\sigma^i = \sigma$ in all scenarios, so that the only scenario quantity in the model is the safety level barrier parameter H.

CDS Calibration with SBAT1P: Scenario Barrier

A Case Study: Vodafone. The Inputs Let us analyze the particular case of the Vodafone telecommunication company at the date of March 10th, 2004 (the data are the mid quotes reported in the Table below. The recovery rate is $\text{Rec} = 40\%$, and the spreads are expressed in basis points).

	CDS maturity T_b	$R_{0,b}^{\text{BID}}(0)$ (bps)	$R_{0,b}^{\text{ASK}}(0)$	$R_{0,b}^{\text{MID}}(0)$
1y	20-mar-05	19	24	21.5
3y	20-mar-07	32	34	33
5y	20-mar-09	42	44	43
7y	20-mar-11	45	53	49
10y	20-mar-14	56	66	61

Maturities of quoted CDS's with their corresponding spreads on March 10th, 2004.

A Case Study: Vodafone. The Inputs

CDS mat	CDS value bid (bps)	CDS value ask (bps)
1y	2.56	-2.56
3y	2.93	-2.93
5y	4.67	-4.67
7y	24.94	-24.94
10y	41.14	-41.14

CDS values computed with deterministic default intensities stripped from mid R quotes but with bid and ask rates R in the premium legs.

We first show how the AT1P time varying model calibrates these data.

In this first example the parameters used for the structural model have been selected by trial and error and based on qualitative considerations, and are $\beta = 0.5$ and $H/V_0 = 0.4$.

We find

A Case Study: Vodafone. The AT1P model calibration

T_b	σ	Surv.	Int.	Surv.
0	32.625%	100.000%	0.357%	100.000%
1y	32.625%	99.625%	0.357%	99.627%
3y	17.311%	98.315%	0.952%	98.316%
5y	17.683%	96.353%	1.033%	96.355%
7y	17.763%	94.206%	1.189%	94.206%
10y	21.861%	89.650%	2.104%	89.604%

Results of the calibrations performed with AT1P vs Deterministic Piecewise linear Intensity.

The calibration is exact and there is no calibration error.

A Case Study: The SBAT1P model calibration. Optimization

Now on to the scenario barrier version SBAT1P. We take $\sigma^i = \sigma = 0.24$.

We start by the first three quotes in the Vodafone Table, but this time calibrate them with an optimization, abandoning the "linear algebra" approach. **We minimize the function expressing the sum of the squares of the CDS's prices in the model, since each CDS should be zero in correspondence of the quoted R.** We solve numerically

$$[H_1^*, H_2^*, p_1^*] = \text{argmin}_{H,p} \sum_{k=1}^{3} [p_1 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}\textsc{gd}; H_1) + p_2 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}\textsc{gd}; H_2)]^2$$

(where $p_2 = 1 - p_1$). We find the following results:

H_i	p_i
0.3188	94.83%
0.6592	5.17%

Target function: practically 0 (exact calibration)

A Case Study: The SBAT1P model calibration. Optimization

In this case the target function is practically zero, so that the calibration is exact. While this could be somehow hoped, since the number of free parameters matches the number of quotes to be calibrated, in general it is not guaranteed.
Let us move to optimizing over the **five quotes** of the Vodafone CDS Table. To this end, we can choose the version with three H scenarios and two probabilities. We run an optimization to solve $[H_1^*, H_2^*, H_3^*, p_1^*, p_2^*] =$

$$= \text{argmin}_{H,p} \sum_{k=1}^{5} \Big[p_1 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_1) + p_2 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_2)$$

$$+ p_3 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_3) \Big]^2$$

where in the optimization we impose the p to take values that are allowed for probabilities.

We obtain the output

H_i	p_i
0.7296	1.24%
0.3384	97.52%
0.7296	1.24%

with target function: $915 bps^2$.

A Case Study: The SBAT1P model calibration. Optimization

H_i	p_i
0.3188	94.83%
0.6592	5.17%

target 0 $\longrightarrow$

H_i	p_i
0.7296	1.24%
0.3384	97.52%
0.7296	1.24%

with target: $915bps^2$.

We will discuss the calibration error shortly. One barrier scenario H remains close to 0.3188, with an even higher probability than before. Notice an important point: the two other scenarios of H more in proximity of default are identical, being both 0.7296, with equal probabilities 0.0124. **This suggests the SBAT1P parametrization to be not effective in describing a dynamics consistent with the CDS cross sectional data we are observing**, since the parameters collapse to a sub-parameterization with just two barrier scenarios (three parameters in total) when fitting five quotes. **This seems to suggest the new parameters we added with respect to the previous case add nothing to explain the increase of data**. And indeed, if one tries the model with two barriers and one probability to fit the five quotes above, consistently with what we just obtained one finds the same calibration error and the parameters:

barrier	probability
0.7296	2.48%
0.3384	97.52%

A Case Study: The SBAT1P model calibration. Optimization

Let us have a look at the calibration error in single CDS's, as implied by the $915bps^2$

CDS maturity T_k	$\text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}, H_1^*, H_2^*, H_3^*; p_1^*, p_2^*, p_3^*)$ (bps)
1y	-2.77
3y	9.99
5y	-1.47
7y	-22.99
10y	16.63

Compare with CDS values obtained inserting R^{bid} and R^{ask} in the premium leg, when using default probabilities stripped by mid R quotes:

CDS mat	CDS value bid (bps)	CDS value ask (bps)
1y	2.56	-2.56
3y	2.93	-2.93
5y	4.67	-4.67
7y	24.94	-24.94
10y	41.14	-41.14

A Case Study: The SBAT1P model calibration. Optimization

The first two CDS are out of the bid ask windows with our calibration, the worst case being the second, giving 9.99 bps of present value against a corresponding bid ask window $[-2.93, 2.93]$.

This situation is not satisfactory, leading to the following extension.

Scenario Barrier and Volatility: SBVAT1P. Numerical Optimization

Now we consider again all of the five quotes of the Vodafone CDS Table and we use the general model with two scenarios on the barrier/volatility parameters (H, ν) and one probability (the other one being determined by normalization to one), with a total of five parameters for five quotes. Here we used $\beta = 0$.

$$[H_1^*, H_2^*; \sigma_*^1, \sigma_*^2; p_1^*] = \text{argmin}_{H,p,\nu} \sum_{k=1}^{5} \Big[p_1 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_1, \sigma^1) +$$

$$+ (1 - p_1) \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_2, \sigma^2) \Big]^2$$

A Case Study: The S$\underline{V}$BAT1P model calibration. Optimization

H_i	σ_i	p_i
0.3721	17.37%	93.87%
0.6353	23.34%	6.13%

We obtain a much lower optimization error than before, i.e. $147bps^2$, corresponding to the following calibration errors on single CDS present values:

T_k	$\text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}, H_{1,2}; \sigma^{1,2}; p_1)$ (bps)
1y	1.38
3y	-3.89
5y	8.16
7y	-7.56
10y	2.41

CDS mat	CDS bid	CDS ask
1y	2.56	-2.56
3y	2.93	-2.93
5y	4.67	-4.67
7y	24.94	-24.94
10y	41.14	-41.14

A Case Study: The S<u>V</u>BAT1P model calibration. Optimization

Now the single CDS calibration errors are much lower than before, being the CDS present values corresponding to market R closer to zero, with the exception of the five years maturity, which could be adjusted by introducing weights in the target function.

We did so by inserting in the objective function weights proportional to the inverse of the bid-ask spread. The results are

H_i	σ_i	p_i
0.3713	17.22%	92.63%
0.6239	22.17%	7.37%

T_k	$\text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}, H_{1,2}; \sigma^{1,2}; p_1)$ (bps)
1y	5.85 (-2.56÷ 2.56)
3y	-3.76 (-2.93÷ 2.93)
5y	4.92 (-4.67÷ 4.67)
7y	-10.46 (-24.94÷ 24.94)
10y	1.47 (-41.14÷ 41.14)

A Case Study: The S$\underline{\text{V}}$BAT1P model calibration. Optimization

Normal Calibration

H_i	σ_i	p_i
0.3721	17.37%	93.87%
0.6353	23.34%	6.13%

Weighted Calibration

H_i	σ_i	p_i
0.3713	17.22%	92.63%
0.6239	22.17%	7.37%

T_k	$\text{CDS}_{0,k}(0, R^{\text{MID}}_{0,k}, H_{1,2}; \sigma^{1,2}; p_1)$ (bps)
1y	1.38 (-2.56÷ 2.56)
3y	-3.89 (-2.93÷ 2.93)
5y	8.16 (-4.67÷ 4.67)
7y	-7.56 (-24.94÷ 24.94)
10y	2.41 (-41.14÷ 41.14)

T_k	$\text{CDS}_{0,k}(0, R^{\text{MID}}_{0,k}, H_{1,2}; \sigma^{1,2}; p_1)$ (bps)
1y	5.85 (-2.56÷ 2.56)
3y	-3.76 (-2.93÷ 2.93)
5y	4.92 (-4.67÷ 4.67)
7y	-10.46 (-24.94÷ 24.94)
10y	1.47 (-41.14÷ 41.14)

The calibration relative to the most traded CDS (3y and 5y) has improved a lot. For the 10y CDS the absolute value is worse than before, but it is well inside the bid-ask spread.

A Case Study: The S<u>V</u>BAT1P model calibration. Optimization

SVBAT1P: volatility scenarios induce a mixture distribution on the firm value:

$$p_{V_t}(x)dx = \mathbb{E}[1_{\{V_t \in dx\}}] = \mathbb{E}\{\mathbb{E}[1_{\{V_t \in dx\}}|H,\sigma]\} = \sum_{i=1}^{N} p_i \mathbb{E}[1_{\{V_t \in dx\}}|H = H_i, \sigma = \sigma^i]$$

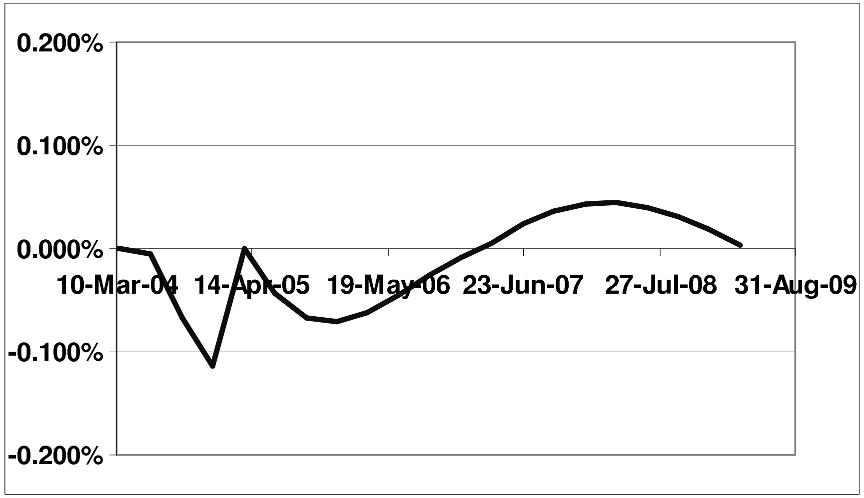
$$= \sum_{i=1}^{N} p_i \, p_{V_t|\sigma=\sigma^i}(x)dx$$

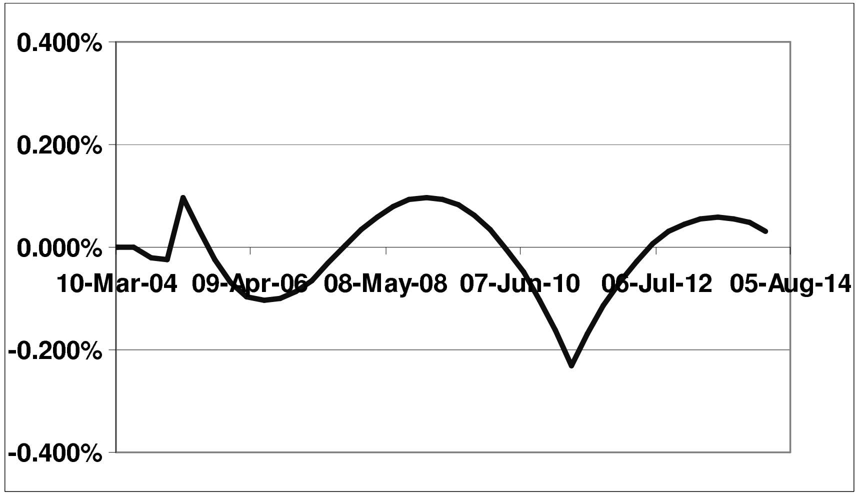
Since the mixture has fat tails, now the trajectories of V spread more easily. One does not need huge volatilities σ, like in the case with no scenario and Gaussian/thin tails returns, to hit the barrier with enough frequency to reproduce high default probabilities.

A Case Study: The SVBAT1P model calibration. Optimization

Difference between SVBAT1P and AT1P survival probabilities

On the left we plot the difference in the case of exact (5y) SBAT1P calibration, on the right the difference in case of (10y) optimization with the SVBAT1P model.





A Case Study: The S<u>V</u>BAT1P model calibration. Optimization

The graphs show that the probabilities with the various models are nearly the same, also in case of non perfect calibration. The main difference between the figures is that in case of exact calibration the probabilities coincide exactly at CDS maturities, while in the optimization case this is not true.

More in general, scenarios on σ and H can be taken jointly (as we did so far) or separately, but one needs to keep the combinatorial explosion under control. The pricing formula remains easy, giving linear combination of formulas in each basic scenario. Taking time-varying parametric forms for the σ^{i}'s can add flexibility.

Brigo and Morini (2006) study the AT1P model with scenarios on the barrier H and a time dependent single scenario volatility. Calibration is perfect in this case.

Counterparty risk in equity return swap (ERS)

Consider an equity return swap. We are a default-free company "A" entering a contract with counterparty "B". The reference underlying equity is of default-free company "C".

"A" and "B" agree on an amount K of stocks of "C" (with price S) to be taken as nominal ($N = K\ S_0$). The contract starts in $T_a = 0$ and has final maturity $T_b = T$.

At $t = 0$ there is no exchange of cash (alternatively, we can think that "B" delivers to "A" an amount K of "C" stock and receives a cash amount equal to KS_0).

At intermediate times "A" pays to "B" the dividend flows of the stocks (if any) in exchange for a periodic rate (e.g. semi-annual LIBOR or EURIBOR L) plus a spread X.

At final maturity $T = T_b$, "A" pays KS_T to "B" (or gives back the amount K of stocks) and receives a payment KS_0.

The price can be derived using risk neutral valuation, and the (fair) spread is chosen in order to obtain a contract whose value at inception is zero.

Counterparty risk in equity return swap (ERS)

Time 0: no flows, or

A $\longrightarrow$ KS_0^C cash $\longrightarrow$ B

A $\longleftarrow$ K equity of "C" $\longleftarrow$ B

Time T_i:

A $\longrightarrow$ equity dividends of "C" $\longrightarrow$ B

A $\longleftarrow$ Libor + Spread $\longleftarrow$ B

Time T_b:

A $\longrightarrow$ K equity of "C" $\longrightarrow$ B

A $\longleftarrow$ KS_0^C cash $\longleftarrow$ B

Counterparty risk in equity return swap (ERS)

We assume the underlying "C" to be default-free, or to have a much stronger credit quality than counterparty "B".

It can be proved that if "B" were default-free itself, the fair spread would be zero. It is then precisely counterparty risk that makes the spread non-zero.

If early default of "B" occurs at time $\tau = \tau_B$, the following happens.

Before τ payments go through normally, as before.

If $\tau \leq T$, compute the net present value (NPV) of the position at time τ.

If NPV is negative to us "A", then at τ we **fully** pay its opposite to "B". Instead, if it is positive to us "A", only a **recovery fraction** REC of that NPV is received.

It is clear the counterparty risk is bad to us "A" when the market is good (large NPV).

Counterparty risk in equity return swap (ERS)

Price of the ERS to us "A":

$$\mathsf{ERS}_0^D = \underbrace{\mathsf{NPV}(0, T_b)}_{0} \quad - \mathrm{L}_{\mathrm{GD}}\mathbb{E}_0\{1_{\{\tau < T_b\}} D(0, \tau) \left(\mathsf{NPV}(\tau, T_b)\right)^+\} \tag{16}$$

where

$$\begin{aligned} \mathsf{NPV}(t, T_b) \quad = \quad & \mathbb{E}_t \Bigg\{ - K \, \mathsf{NPV}_{dividends}^{t,T_b}(t) + K S_0 \sum_{i=\beta(t)}^{b} D(t, T_i) \alpha_i \left(L(T_{i-1}, T_i) + X \right) \\ & + \left(K S_0 - K S_{T_b} \right) D\left(t, T_b\right) \Bigg\}. \end{aligned} \tag{17}$$

We denoted by $\mathsf{NPV}_{dividends}^{s,t}(u)$ the net present value of the dividend flows between s and t computed in u.

Counterparty risk in equity return swap

A **key variable** for the valuation is the **correlation** ρ between counterparty V^B and underlying equity S^C.

Recall: Firm Value of the counterparty "B" (AT1P or S(V)BAT1P) Calibrated to CDS of "B" and underlying equity of name "C" (Black Scholes) with $\sigma_C = 20\%, \quad q = 0.8\%$ and r given by the zero curve:

$$dV(t) = V(t)\,r(t)\,dt + V(t)\,\sigma_V(t)\,dW_B(t)$$

$$dS_C(t) = S_C(t)\,(r(t) - q(t))\,dt + S_C(t)\,\sigma_C(t)\,dW_C(t)$$

$$\text{Correlation:} \quad dW_B dW_C = \rho_{B,C} dt$$

can be estimated as historical correlation between daily returns of equities "B" and "C" or a view on it can be expressed.

We can find the value of X that makes the contract fair by iterating the MC simulation until the payoff value is sufficiently small.

Counterparty risk in equity return swap: AT1P model

AT1P model with $\sigma(\cdot)$ calibrated to Vodafone CDS as seen earlier

ρ	X	ES payoff	MC error
-1	0	0	0^*
-0.2	2.45	-0.02	1.71
0	4.87	-0.90	2.32
0.5	14.2	-0.53	2.71
1	24.4	-0.34	0.72

Fair spread X (in basis pts) of the ERS for 5 different correlations. Maturity $T = 5y$.

We also report the average of the simulated payoff (times 10000 on a notional $S_0 = 20$) across the 2000000 scenarios and its standard error thus showing that X is indeed fair, since averages and MC windows are all close to zero.

The MC error for $\rho = -1$ $(*)$ is null since in each scenario the simulated term has been found to be zero (i.e. the NPV was always negative).

Counterparty risk in equity return swap: SBAT1P vs AT1P

We now insert the fair AT1P-X into the payoff and value it with SBAT1P calibrated to the same CDS data. We thus find the different implications of the two models, calibrated to the same data, when valuing counterparty risk in equity payoffs. We find

ρ	X	ES payoff	MC error estimate
-0.2	2.45	-28.44	1.49
0	4.87	3.45	2.04
0.5	14.2	165.50	2.29

Recalling that $S_0 = 20$, for a unit notional we would get a max discrepancy about $165/20 \approx 8$ bps. By comparing with the bid-offer window induced by market bid-offer R's on CDS's with unit notional, as shown earlier we realize that this difference is small.

The change in X of AT1P that would match the 165.5 bps payoff value is of about 2 bps on the $X = 14.2$ bps value.

Counterparty risk in ERS: SVBAT1P vs SBAT1P vs AT1P

With S**V**BAT1P above calibrated to the same data in the case $\rho = 0.5$, the MC method gives us the payoff expected value as 292.03 bps, with a MC error of 1.67 bps.

Since $S_0 = 20$ we consider $292.03/20 \approx 14.6$, and compare with CDS payoff bid ask values. We see that we are within the 7y and 10y CDS bid ask spreads but not within the 1y, 3y and 5y ones.

We find that a spread of $X = 17.3$ in AT1P with $\rho = 0.5$ would give an expected payoff value of 289, so that we see that the difference between AT1P and SVBAT1P in terms of AT1P spread X is of about 17.3-14.2= 3.1 bps on $X = 14.2$.

The differences are relevant but contained: the fair spread depends on the model but different models calibrated to the same data give the same order of magnitude for X

Hints at Equity Default Swaps with SVBAT1P

Recall EDS: An EDS is typically a 5 years maturity OUT OF THE MONEY AMERICAN DIGITAL PUT struck at 30% of the spot price.

There is "default" if the equity, before 5 years, falls below 30% of its initial price.

The main difference between EDS and CDS is in the variable triggering default. If we adopt a first passage model for credit derivatives for example, like our AT1P model or the SVBAT1P model, it is the firm value V and not the firm equity S that triggers default. In structural models, as we have seen above, S in general is a *barrier option* on V.

It turns out EDS are not a purely equity product. Why so, if the payoff is equity and if the price can be in principle computed by means of an equity smile model? What has credit to do with EDS??

Hints at Equity Default Swaps with SVBAT1P

The problem is in the calibration data for the equity smile model.

There are no implied vol data from the equity option market at low strikes $30\% S_0$.

The only products that may contain data on that region of the underlying are CDS.

Equity Default Swaps

The idea is: use the SVBAT1P model to link the firm value V and the firm equity S (see also Jones et al (1984) and above Hull, Nelken and White (2004)). Find an expression for the debt (and thus the equity S) within the chosen structural model V.

Equity expressions are known in closed form for time-constant and standard barrier Black Cox models (e.g. Bielecki and Rutkowski (2001), Chapter 3).

Under SVBAT1P we simply obtain a linear combination of said formulas under each scenario, weighted by scenarios probabilities.

Then we can price an EDS by means of MC simulation of the SVBAT1P V, from which, scenario by scenario, we deduce analytically the equity S value by means of the found formula.

Equity Default Swaps

This is currently under investigation. In algorithmic form:

- Calibrate the parameters of the SVBAT1P dV model to CDS
- Compute the equity as an option on V: $S_t = \text{Option}(t, V_t)$ (linear combination of formulas in Bielecki and Rutkowski (2001), Chapter 3)
- Simulate SVBAT1P V (easy) and through formula in the previous point deduce paths for S
- Price the EDS with the resulting S paths

This way we price the EDS with an equity model entirely calibrated to credit data.

Equity Default Swaps

Albanese and Chen (2004), in

`http://defaultrisk.com/pp_crdrv_66.htm`

suggest currently the EDS prices observed in the market in 2005 are consistent with a CEV local volatility model: in this model the equity is modeled directly as

$$dS_t = r(t) S_t dt + \sigma \ S_t^{\gamma} dW_t$$

For $\gamma < 1$ there is absorbtion in 0. Albanese and Chen model the CDS default as the time when S hits zero first. They suggest the market quotes reflect these dynamics, even if this local volatility model is missing important features, such as jumps.

They propose different, more appropriated models that, however, do not reproduce market prices for the time being. For more details we refer to their original work.

Other Structural Models

One of the main perceived drawbacks of Structural Models is that they cannot achieve short term credit spreads. This is due to the particular lognormal (diffusion) dynamics chosen for the underlying: If there is a finite distance from the barrier, the continuous process cannot touch it in the next infinitesimal interval.

Formally the short term credit spread $\lambda(t)$ is the difference between the short rate $\bar{r}(t)$ of a risky bond and the risk free short rate $r(t)$. It can be computed as

$$\lambda(t) = \lim_{dt \to 0} \frac{\mathbb{Q}\{\tau \in [t, t+dt) | \mathcal{F}_t \wedge \{\tau \geq t\}\}}{dt}$$

What actually happens is that in the market there are no sources for very short credit spreads, on maturities smaller than 3/6 months. And to reproduce the shortest observable spreads, it is sufficient to calibrate the structural model to $3m/6m$ maturities to obtain reliable credit spreads.

Nonetheless, there are a lot of models trying to achieve non-null credit spreads on very short (unobservable) maturities, mostly for the realism of hedging strategies.

CreditGrades Model

Possible ways to mathematically explain short term credit spreads:

- Jump-diffusion (Zhou (2001)): a jump component is added to the underlying continuous process. It has the drawback to be a somehow untractable model.
- Uncertainty on the firm value (Duffie and Lando (1997))
- Uncertainty on the value of the underlying barrier (Giesecke (2002), Finkelstein, Lardy et al. (2002)) that is modeled as a continuous random variable (unlike our SVBAT1P). The Finkelstein and Lardy model has been copyrighted under the name of CreditGrades

Here we just have a look at the CreditGrades model

CreditGrades Model

Continuous lognormal underlying process with constant coefficients for the firm value and time-constant barrier (a possible Black and Cox framework). The new idea is that the investor does not have a precise idea on the exact barrier, that is then considered random.

The model assumes a value of the firm V and a lognormal shape for the barrier B as:

$$dV = \sigma_V V\, dW, \qquad B = \bar{B} e^{\lambda Z - \lambda^2/2}$$

where Z is a standard normal variable independent on the firm value V. The default time:

$$\tau = \inf\{t : \ V(t) \leq B\}$$

and the survival probability can be computed through the approximated formula

$$\mathbb{Q}\{\tau > t\} = \Phi\left(\frac{b\lambda}{\bar{\sigma}_t} - \frac{\bar{\sigma}_t}{2}\right) - e^{\lambda b}\Phi\left(-\frac{b\lambda}{\bar{\sigma}_t} - \frac{\bar{\sigma}_t}{2}\right)$$

where $b = \lambda + \frac{1}{\lambda}\ln\frac{V_0}{\bar{B}}$ and $\bar{\sigma}_t^2 = \sigma_V^2 t + \lambda^2$.

CreditGrades Model

Actually both V_0 and σ_V are not observable. Some approximation is needed. For the initial value V_0 they use $V_0 = S_0 + \bar{B}$ where S_0 is the initial value of the equity and $\bar{B}$ is the reference value for the barrier.

Then an approximation for the volatility is needed: According to Merton's Model

$$\sigma_S = \sigma_V \Delta_{\text{call}} \frac{V}{S}$$

We can imagine that at $t = 0$ the company is well-behaving, so that $V/B \gg 1$, then $V \simeq S$ and then $\Delta_{\text{call}} = \frac{\partial S}{\partial V} \simeq 1$. Moreover we can approximate $V \simeq S + \bar{B}$ obtaining

$$\sigma_S = \sigma_V \frac{S + \bar{B}}{S}$$

CreditGrades Model

$$\sigma_S = \sigma_V \frac{S + \bar{B}}{S}$$

This corresponds to use a displaced diffusion coefficient in the equity dynamics.

$$dS = \sigma_V (S + \bar{B}) dW$$

This is important since a displaced diffusion shifts the distribution.

It is a model commonly used for the volatility skew.

Indeed we see that this relation can explain the volatility skew in the equity options market. In fact it implies a decreasing dependence on S for the volatility, a pattern which is actually observable in the market.

Structural Models: Link with Equity

Usually some positive dependence between CDS spreads and the equity vol σ_S of the reference entity is observed.

Attempts to deduce credit spreads from equity data have been made in the past.

$$\sigma_S \to \sigma_V = \text{function}(\sigma_S) \to \text{CDS}(V, \sigma_V) \to \mathbb{Q}(\tau > T_i) \to \text{Credit Spreads}$$

All of them are based on the same ideas: Get the equity volatility σ_S from the market, deduce σ_V using the Merton model approximation based on Δ, with this σ_V compute the CDS price and survival probabilities, from which you have credit spreads.

Results are not always good, since these procedures imply several approximations. Since now CDS are rather liquid, this procedure has little use.

Structural Models: Link with Equity

It would be more useful perhaps to look for a better specification of the debt structure, and possibly for a better specification of the link between σ_S and σ_V.

In this case we could proceed in the opposite direction:

CDS Quotes $\longrightarrow$ structural model $\longrightarrow \sigma_V \longrightarrow \sigma_S =$ function$(\sigma_V) \longrightarrow$ EDS pricing

Starting from CDS spreads we could derive estimates for σ_V and then for σ_S.

This approach could be useful for example when pricing Equity Default Swap contracts, i.e. deep out of the money put, for which there is no quoted implied volatility for the levels of strike that are relevant for EDS. We have hinted at this procedure above under the SVBAT1P model.

Conclusions on Structural models

We introduced basic structural models: Merton and Black Cox. We explained the different philosophy with respect to intensity models.

We illustrated how our tractable first passage structural model AT1P is calibrated exactly to CDS data and showed some case studies based on market data when the credit quality of the underlying name is deteriorating.

We extended the AT1P structural model using random parameters in the dynamics and in the safety covenants barrier, leading to the S(V)BAT1P models, and studied what changes in the calibration and the opportunities given by the new models.

We shortly analyzed the application of the two models to counterparty risk pricing in an equity return swap.

We explored possible links between value of the firm models and equity models, taking into account the equity volatility smile

We hinted at equity default swap valuation under some of our structural models

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