Credit and Default Modeling

# UNIT 2

# SINGLE NAME MODELS: REDUCED FORM

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# UNIT 2. SINGLE NAME MODELS: REDUCED FORM

- **Modeling Tools: Poisson Processes**;
- Time inhomogeneous Poisson Processes: deterministic credit spread (intensity);
- Stochastic intensity Poisson Processes (Cox processes, credit spread volatility).
- Two Important Technicalities: **i) dependence default-interest rates** and **ii) The filtration switching formula**
- CDS Market implied credit spread (deterministic intensity);
- A clever numeraire and the CDS options market model (embedded stochastic intensity);
- First examples of implied CDS rates volatilities;
- Explicit stochastic intensity modeling: The SSRD Model;
- SSRD Analytic and Automatic calibration to CDS market data and interest rate data;
- CDS options with the SSRD model (with CIR++ stochastic intensity)
- Relationship between CIR++ parameters and implied CDS volatilites;

# The Basic Idea of Reduced Form Models

In reduced form or intensity models, the default time $\tau$ obeys roughly the following:

Having not defaulted before $t$, Probability of defaulting in the next $dt$ instants is

$$\text{Prob}(\tau \in [t, t + dt) | \tau > t, \text{market info up to } t) = \lambda(t)dt$$

where the "probability" $dt$ factor $\lambda$ is called **intensity** or **hazard rate**. It is also an **instantaneous credit spread** (more on this later). Intensity can be

- Constant ($\tau$ is first jump of time homogeneous Poisson process);
- Time varying ($\tau$ is first jump of time inhomogeneous Poisson process); Can model the term structure of credit spreads; Does not model credit spread volatility; Implied hazard functions;
- Stochastic ($\tau$ is first jump of Cox Process); Can model term structure of credit spreads; Can model credit spread volatility;

# Modeling Tools: Poisson processes

DEFINITION. A "Time homogeneous Poisson process" is a **unit-jump increasing, right continuous process** with **stationary independent increments** and $M_0 = 0$.

From this definition it follows that
**LEMMA**. $M$ time homogeneous Poisson process. Then
1) exists positive $\bar\gamma$ such that $\mathbb{Q}\{M_t = 0\} = \exp(-\bar\gamma t)$ for all $t$.
(proof: if $f(t) := \mathbb{Q}\{M_t = 0\}$ then $f(t+s) = f(t)f(s)$ and then...)
2) $\lim_{t\to 0} \mathbb{Q}\{M_t \geq 2\}/t = 0$ (proof of this is more complicated)
3) $\lim_{t\to 0} \mathbb{Q}\{M_t = 1\}/t = \bar\gamma$

**THEOREM**. $M$ time homogeneous Poisson process. Then

$$\mathbb{Q}\{M_t - M_s = k\} = e^{-\bar\gamma(t-s)}(\bar\gamma(t-s))^k/k!$$

i.e. $M_t - M_s \sim$ PoissonRandVar$((t-s)\bar\gamma)$, with $M_t - M_s$ independ of $\sigma(\{M_u, u \leq s\})$.

# Modeling Tools: Poisson processes

**CHARACTERIZATION**. $M$ time-homogeneous Poisson process with param $\bar{\gamma} \iff$
$\iff \mathbb{E}[M_{t+s} - M_t | M_u : u \le t] = \bar{\gamma} s$ for all $t, s > 0$.

**INTERPRETATION OF** $\bar{\gamma}$. $M$ time-homogeneous Poisson process. Then
$\bar{\gamma} = \mathbb{E}(M_t)/t = \mathsf{VAR}(M_t)/t$ (average arrival rate)

**THEOREM**. $M$ time homogeneous Poisson process. Let $\tau^1, \tau^2, \ldots, \tau^m, \ldots$ be the first, second etc. jump times of $M$. **Then** $\tau^1, \tau^2 - \tau^1, \tau^3 - \tau^2, \ldots$,
i.e. the times between one jump and the next one, are i.i.d. $\sim$ exponentialRandVar($\bar{\gamma}$)

( EQUIVALENTLY, the R.V's $\bar{\gamma}\tau^1, \bar{\gamma}(\tau^2 - \tau^1), \bar{\gamma}(\tau^3 - \tau^2)$.. are i.i.d. $\sim$ expRandVar$(1)$)

Important Consequence: $\mathbb{Q}\{\bar{\tau} \in [t, t + dt) | \bar{\tau} \ge t\} = \bar{\gamma}\, dt$

In the simplest intensity model **the default time is modeled as** $\tau^1$.

# Modeling Tools: Poisson processes. Default as first jump

Summing up, if we define the default time as $\bar\tau := \tau^1$, the first jump time of $M_t$, then $\bar\tau$ has the following properties: $\bar\gamma\ \bar\tau \sim \text{exponentialRandVar}(1)$, independent of $\mathcal{F}$, and we have $\mathbb{Q}\{\bar\tau > t\} =$

$$= \mathbb{Q}\{\bar\gamma\bar\tau > \bar\gamma t\} = \mathbb{Q}\{\text{exponentialRV}(1) > \bar\gamma t\} = e^{-\bar\gamma t} \Rightarrow$$

$$\Rightarrow \mathbb{Q}\{\bar\tau \in [t, t+dt) | \bar\tau \geq t\} = \frac{\mathbb{Q}\{\bar\tau \in [t, t+dt) \bigcap \bar\tau \geq t\}}{\mathbb{Q}\{\bar\tau > t\}} = \frac{\mathbb{Q}\{\bar\tau \in [t, t+dt)\}}{\mathbb{Q}\{\bar\tau > t\}}$$

$$= \frac{\mathbb{Q}\{\bar\tau > t\} - \mathbb{Q}\{\bar\tau > t + dt\}}{\mathbb{Q}\{\bar\tau > t\}} = \frac{e^{-\bar\gamma t} - e^{-\bar\gamma(t+dt)}}{e^{-\bar\gamma t}} = \bar\gamma\ dt$$

for small $dt$; **"probability that company defaults in (arbitrarily small)** "$dt$" **years given that it has not defaulted so far is $\bar\gamma\ dt$."** Also, prob of defaulting between $s$ and $t$ is

$$\mathbb{Q}\{s < \bar\tau \leq t\} = \exp(-\bar\gamma s) - \exp(-\bar\gamma t) \approx \bar\gamma(t - s)$$

# Poisson processes. Default intensity as credit spread

We have just seen that, if $\bar{\tau} := \tau^1$ is the first jump time of a Poisson process $M_t$ with intensity $\bar{\gamma}$, then

$$\mathbb{Q}\{\bar{\tau} > t\} = e^{-\bar{\gamma}t}, \quad \mathbb{Q}\{\bar{\tau} \in [t, t+dt) | \bar{\tau} \geq t\} = \bar{\gamma}\, dt.$$

The first formula is very important. It tells us that **survival probabilities have the same structure as discount factors, with the default intensity playing the role of interest rates**. This is an extremely important consequence of the exponential distribution for times between jumps.

**It is this fundamental property of jumps of poisson processes that allows us to see survival probabilities as discount factors, and thus default intensities as credit spreads. This allows us to use much of the interest-rate technology in default modeling under these kind of reduced form models**

# Modeling Tools: time inhomogeneous Poisson Processes

We consider now **deterministic time-varying** intensity $\gamma(t)$, which we assume to be a positive and piecewise continuous function. We define

$$\Gamma(t) := \int_0^t \gamma(u)du,$$

the **cumulated intensity, cumulated hazard rate,** or also **Hazard function**.

If $M_t$ is a Standard Poisson Process, i.e. a Poisson Process with intensity one, than a **time-inhomogeneous Poisson Process** $N_t$ with intensity $\gamma$ is defined as

$$N_t = M_{\Gamma(t)}.$$

So a time inhomogeneous PP is just a time-changed PP.

$N_t$ is still increasing by jumps of size $1$, its increments are still independent, but they are **no longer identically distributed** due to the "time distortion" introduced by $\Gamma$.

## Modeling Tools: time inhomogeneous Poisson Processes

From $N_t = M_{\Gamma(t)}$ We have obviously

$N$ jumps the first time at $\tau \iff M$ jumps the first time at $\Gamma(\tau)$.

But since we know that $M$ is standard Poisson Process for which first jump time is exponentialRV(1), then

$$\Gamma(\tau) =: \xi \sim \text{exponentialRandVar}(1)$$

By inverting this last equation we have that

$$\tau = \Gamma^{-1}(\xi),$$

with $\xi$ standard exponential random variable.

# Time inhomogeneous Poisson Pr: Time varying credit spreads

Also, we have easily

$$\mathbb{Q}\{s < \tau \leq t\} = \mathbb{Q}\{\Gamma(s) < \Gamma(\tau) \leq \Gamma(t)\} = \mathbb{Q}\{\Gamma(s) < \xi \leq \Gamma(t)\} =$$

$$= \mathbb{Q}\{\xi > \Gamma(s)\} - \mathbb{Q}\{\xi > \Gamma(t)\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)) \text{ i.e.}$$

**"prob of default between $s$ and $t$ is** "$\mathbf{e}^{-\int_0^s \gamma(\mathbf{u})\mathbf{du}} - \mathbf{e}^{-\int_0^t \gamma(\mathbf{u})\mathbf{du}} \approx \int_s^t \gamma(\mathbf{u})\mathbf{du}$" (where the final approximation is good for small exponents). It is easy to show, along the same lines, that

$$\mathbb{Q}\{\tau \in [t, t+dt) | \tau \geq t\} = \gamma(t)\ dt.$$

**"probability that company defaults in (arbitrarily small) "$dt$" years given that it has not defaulted so far is $\gamma(t)\ dt$."**

## Time inhomogeneous Poisson Pr: Time varying credit spreads

$$\Gamma(t) := \int_0^t \gamma(u)du, \quad \tau = \Gamma^{-1}(\xi) \Rightarrow$$

$$\Rightarrow \quad \mathbb{Q}\{\tau > t\} = e^{-\int_0^t \gamma(u)du}, \quad \mathbb{Q}\{\tau \in [t, t+dt) | \tau \geq t\} = \gamma(t)\ dt.$$

**Again, the Poisson Process core structure with exponentially distributed in-between-jump times is allowing us to see survival probabilities as discount factors, and thus default intensities as credit spreads. This allows us to use much of the interest-rate technology in default modeling**

But...

$\xi$ **is independent of all default free market quantities and represents an external source of randomness that makes reduced form model incomplete** (more on this later),

# Time inhomogeneous Poisson Pr: Time varying credit spreads

Here the time-varying nature of $\gamma$ allows us to take into account the TERM STRUCTURE OF CREDIT SPREADS. It is actually this model that is used to strip default probabilities from CDS quotes, for example.

This formulation does not take into account CREDIT SPREAD VOLATILITY, since $\gamma$ is deterministic and we have

$$d\gamma(t) = (\ldots)dt + \boxed{0}\, dW_t.$$

# Modeling Tools: Cox processes

Intensity can be also time-varying and **stochastic**: in that case it is assumed to be at least a $\mathcal{F}_t$-adapted and right continuous (and thus progressive) process and is denoted by $\lambda_t$ and the **cumulated intensity** or **hazard process** is the random variable $\Lambda(T) = \int_0^T \lambda_t dt$. We assume $\lambda_t > 0$.

"$\mathcal{F}_t$-Adapted" means that given $\mathcal{F}_t$, i.e. the default-free market info up to time $t$, we know $\lambda$ from $0$ to $t$.

In a **Cox process** with stochastic intensity $\lambda$, conditional on $\mathcal{F}^\lambda$ (i.e. on $\lambda$), we still have a Poisson Process structure, **and all facts we have seen for the case with** $\gamma(t)$ **still hold, conditional on** $\lambda$.

Under stochastic intensity, the final process jumping first at $\tau$ is called a **Cox process.**

We have that, for Cox processes, default is defined as $\tau := \Lambda^{-1}(\xi)$.

Notice that here not only $\xi$ is random (and still independent of anything else, included $\lambda$), but $\lambda$ itself is stochastic. This is why Cox processes are at times called "doubly stochastic Poisson processes".

# Modeling Tools: Cox processes

$$\Lambda(t) = \int_0^t \lambda_u du \quad , \quad \tau := \Lambda^{-1}(\xi),$$

with $\lambda$ positive and $\mathcal{F}_t$ right continuous and adapted.
We have $\mathbb{Q}\{\tau \in [t, t+dt)|\tau \geq t, \mathcal{F}_t\} = \lambda_t \ dt$. This reads, if "$t$=now":
**"probability that company defaults in (small) "$dt$" years given that it has not defaulted so far and given the default-free-market information so far is $\lambda_t \ dt$."**

Under standard assumptions one can show $\mathbb{Q}\{\tau \geq s\} =$

$$= \mathbb{Q}\{\Lambda(\tau) \geq \Lambda(s)\} = \mathbb{Q}\left\{\xi \geq \int_0^s \lambda(u)du\right\} = \mathbb{E}\left[\mathbb{Q}\left\{\xi \geq \int_0^s \lambda(u)du \middle| \mathcal{F}^\lambda\right\}\right] = \mathbb{E}\left[e^{-\int_0^s \lambda(u)du}\right]$$

which is completely analogous to the bond price formula in a short rate model with interest rate $\lambda$. **Cox processes allow to drag the interest-rate technology and paradigms into default modeling**. But again...

**$\xi$ is independent of all default free market quantities (of $\mathcal{F}$, of $\lambda$...) and represents an external source of randomness that makes reduced form models incomplete.**

# Cox processes: Term structure of Credit Spreads AND their volatility

Now the time varying nature of $\lambda$ may account for the term structure of credit spreads, while the stochasticity of $\lambda$ can be used to introduce credit spread volatility. For example, in a (jump-?) diffusion setting,

$$d\lambda_t = b(t, \lambda_t)dt + \sigma(t, \lambda_t)\ dW_t(+dJ_t?)$$

We will see some explicit examples like SSRD and CIR++ later on.

## Modeling Tools: Cox processes

Summing up:

SPP : Standard Poisson Processes (with unit constant intensity, i.e. instantaneous jump probability) are the probabilistic basis; **Intensity** (or instantaneous credit spread) **is constant** and set to one.

PP : Time inhomogeneous Poisson processes can be built based on SPP and on a given **deterministic time-varying intensity**; these are often used as a quoting mechanism for credit spreads in **CDS and Corporate Bond** contracts, the intensity being also interpreted as an **instantaneous credit spread**;

COX : If the intensity is Stochastic, conditional on the intensity filtration we have a PP and this is a Cox process. These models can be used for more sophisticated credit derivatives and take into account also **credit spread volatility.**

# CDS Calibration and Implied Hazard Rates/Intensities

Reduced form models are the models that are most commonly used in the market to infer implied default probabilities from market quotes.

Market instruments from which these probabilities are drawn are especially CDS and Bonds.

We will see in some detail the procedure concerning CDS.

The reduced-from model used for this is the time-inhomogeneous Poisson Process, with time varying intensity $\gamma(t)$ and cumulated intensity/hazard function $\Gamma(t) = \int_0^t \gamma(u)du$.

# Time inhomogeneous Poisson Process: Examples of Hazard rates stripping from CDS quotes

Recall: Having not defaulted by $t$, Probability of defaulting in the next $dt$ instants is

$$\text{Prob}(\tau \in [t, t+dt) | \tau > t, \text{market info up to } t) = \gamma(t) dt$$

where the "probability" $dt$ factor $\gamma$ is called **intensity** or **hazard rate**. It is also an **instantaneous credit spread**. Probability of surviving $t$ is

$$\text{Prob}(\tau > t) = \exp\left(-\int_0^t \gamma(s) ds\right), \quad \text{Prob}(\tau \in dt) = \gamma(t) \exp\left(-\int_0^t \gamma(s) ds\right) dt$$

Here we take the hazard rate $\gamma$ to be deterministic and **piecewise constant**:

$$\gamma(t) = \gamma_i \quad \text{for} \quad t \in [T_{i-1}, T_i). \quad (\gamma_1, \gamma_2, \ldots, \gamma_i, \ldots)$$

Notice that $\Gamma(t) = \int_0^t \gamma(s) ds = \sum_{i=1}^{\beta(t)-1} (T_i - T_{i-1})\gamma_i + (t - T_{\beta(t)-1})\gamma_{\beta(t)}$

# Examples of Hazard rates stripping from CDS quotes

Set $\Gamma_j := \int_0^{T_j} \gamma(s)ds = \sum_{i=1}^{j}(T_i - T_{i-1})\gamma_i$

In this context, compute for example the protection leg term of a CDS:

$$\mathrm{L}_{\mathrm{GD}}\, \mathbb{E}[D(0,\tau)\mathbf{1}_{\{T_a<\tau<T_b\}}] = \mathrm{L}_{\mathrm{GD}} \int_0^\infty \mathbb{E}[D(0,u)\mathbf{1}_{\{T_a<u<T_b\}}]\mathbb{Q}(\tau \in du)$$

$$= \mathrm{L}_{\mathrm{GD}} \int_{T_a}^{T_b} \mathbb{E}[D(0,u)]\mathbb{Q}(\tau \in du) = \mathrm{L}_{\mathrm{GD}} \int_{T_a}^{T_b} P(0,u)\gamma(u)\exp\left(-\int_0^u \gamma(s)ds\right)du$$

$$= \mathrm{L}_{\mathrm{GD}} \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1}))P(0,u)du$$

# Examples of Hazard rates stripping from CDS quotes

With similar computations for the other CDS terms, it can be shown that under this formulation we have

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \Gamma(\cdot)) = \mathbf{1}_{\{\tau > t\}} \left[ R \int_{T_a}^{T_b} P(t,u)(T_{\beta(u)-1} - u) d(e^{-(\Gamma(u)-\Gamma(t))}) \right.$$

$$\left. + \sum_{i=a+1}^{b} P(t, T_i) R \alpha_i e^{-(\Gamma(T_i)-\Gamma(t))} + \mathrm{L}_{\mathrm{GD}} \int_{T_a}^{T_b} P(t,u) d(e^{-(\Gamma(u)-\Gamma(t))}) \right]$$

and in particular

$$\mathrm{CDS}_{a,b}(0, R, \mathrm{L}_{\mathrm{GD}}; \Gamma(\cdot)) = \left[ R \int_{T_a}^{T_b} P(0,u)(T_{\beta(u)-1} - u) d(e^{-\Gamma(u)}) \right.$$

$$\left. + \sum_{i=a+1}^{b} P(0, T_i) R \alpha_i e^{-\Gamma(T_i)} + \mathrm{L}_{\mathrm{GD}} \int_{T_a}^{T_b} P(0,u) d(e^{-\Gamma(u)}) \right]$$

# Examples of Hazard rates stripping from CDS quotes

So that $\mathrm{CDS}_{a,b}(0, R, \mathrm{L}_{\mathrm{GD}}; \Gamma(\cdot)) =$

$$= \Bigg[ R \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1}))P(0,u)(u - T_{i-1})du$$

$$+R \sum_{i=a+1}^{b} P(0,T_i)\alpha_i e^{-\Gamma(T_i)} - \mathrm{L}_{\mathrm{GD}} \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1}))P(0,u)du]$$

# Examples of Hazard rates stripping from CDS quotes

Now in the market $T_a = 0$ and we have fair $R$ quotes for $T_b = 1y, 2y, 3y, \ldots, 10y$, with $T_i$'s resetting quarterly.

$\mathrm{CDS}_{0,b}(0, R, \mathrm{L}_{\mathrm{GD}}; \Gamma(\cdot)) =$

$$= \Big[ R \sum_{i=1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1})) P(0,u)(u - T_{i-1}) du$$

$$+ R \sum_{i=1}^{b} P(0,T_i)\alpha_i e^{-\Gamma(T_i)} - \mathrm{L}_{\mathrm{GD}} \sum_{i=1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1})) P(0,u) du]$$

We solve

$$\mathrm{CDS}_{0,1y}(0, R_{0,1y}^{MKT}, \mathrm{L}_{\mathrm{GD}}; \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 =: \gamma^1) = 0;$$

$$\mathrm{CDS}_{0,2y}(0, R_{0,2y}^{MKT}, \mathrm{L}_{\mathrm{GD}}; \gamma^1; \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 =: \gamma^2) = 0; \ldots.$$

## Examples of Hazard rates stripping from CDS quotes

September 10th, 2003 Recovery Rate = 40%

| Maturity $T_b$ (yr) | Maturity (dates) | $R_{0,b}$ |
|---|---|---|
| 1 | 20-Sep-04 | 192.5 |
| 3 | 20-Sep-06 | 215 |
| 5 | 20-Sep-08 | 225 |
| 7 | 20-Sep-10 | 235 |
| 10 | 20-Sep-13 | 235 |

Table 2: Maturity dates and corresponding CDS quotes in bps for September 10th, 2003.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 10-Sep-03 | 3.199% | 100.000% |
| 20-Sep-04 | 3.199% | 96.714% |
| 20-Sep-06 | 4.388% | 89.552% |
| 22-Sep-08 | 3.659% | 82.508% |
| 20-Sep-10 | 5.308% | 75.357% |
| 20-Sep-13 | 2.338% | 67.078% |

Table 3: Calibration with piecewise linear intensity on September 10th, 2003.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 10-Sep-03 | 3.199% | 100.000% |
| 20-Sep-04 | 3.199% | 96.714% |
| 20-Sep-06 | 3.780% | 89.578% |
| 22-Sep-08 | 4.033% | 82.516% |
| 20-Sep-10 | 4.458% | 75.402% |
| 20-Sep-13 | 3.891% | 66.978% |

Table 4: Calibration with piecewise constant intensity on September 10th, 2003.

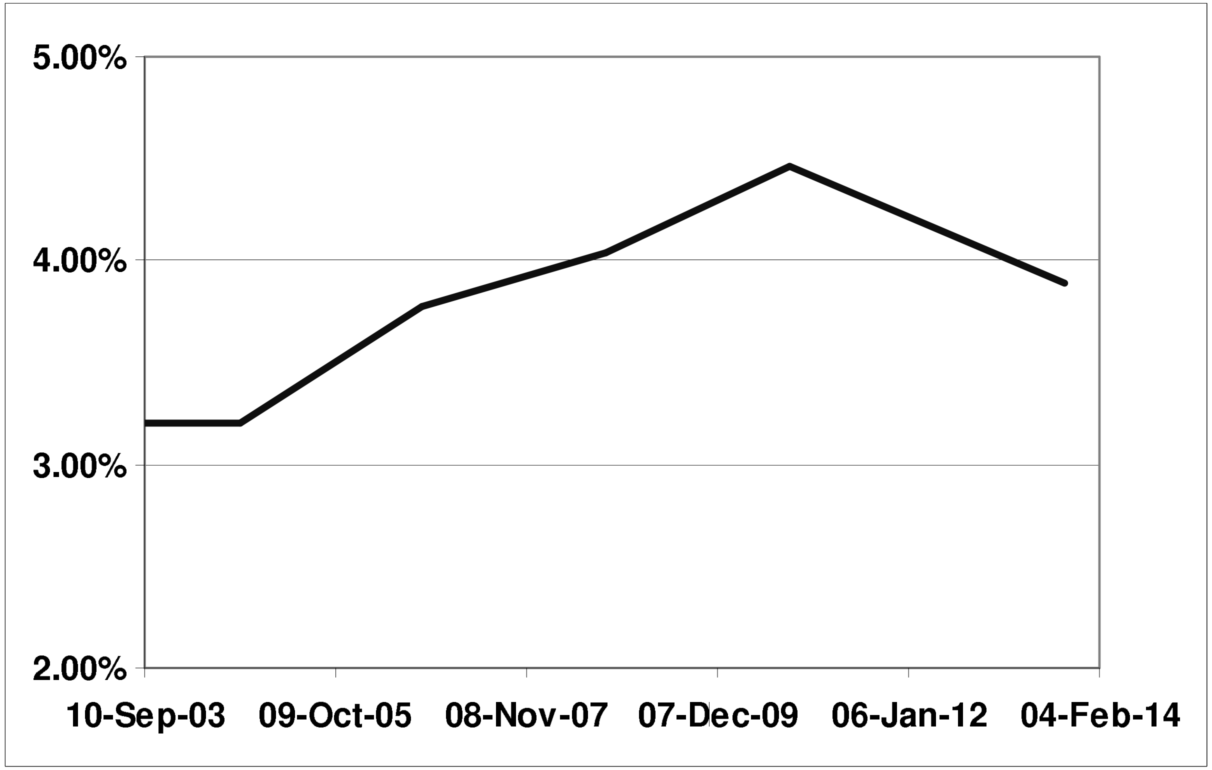


Figure 5: Piecewise linear intensity $\gamma$ calibrated on CDS quotes on September 10th, 2003.

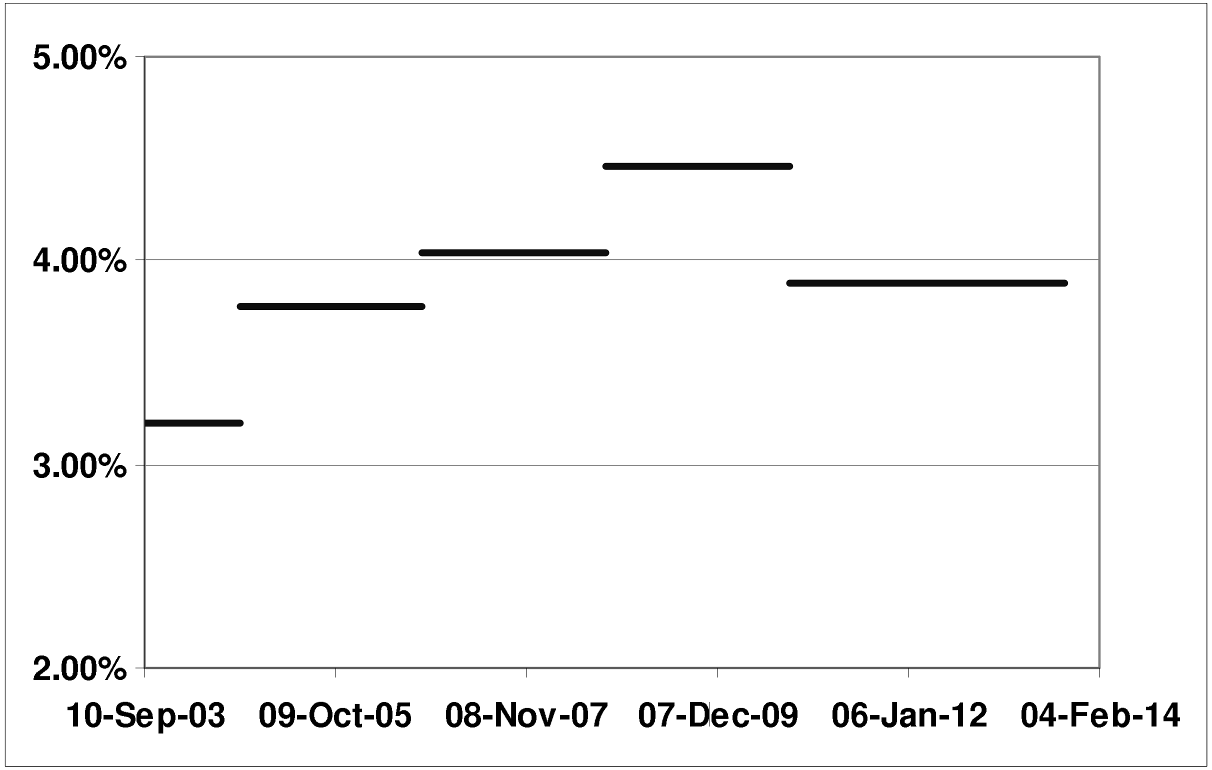


Figure 6: Piecewise constant intensity $\gamma$ calibrated on CDS quotes on September 10th, 2003.

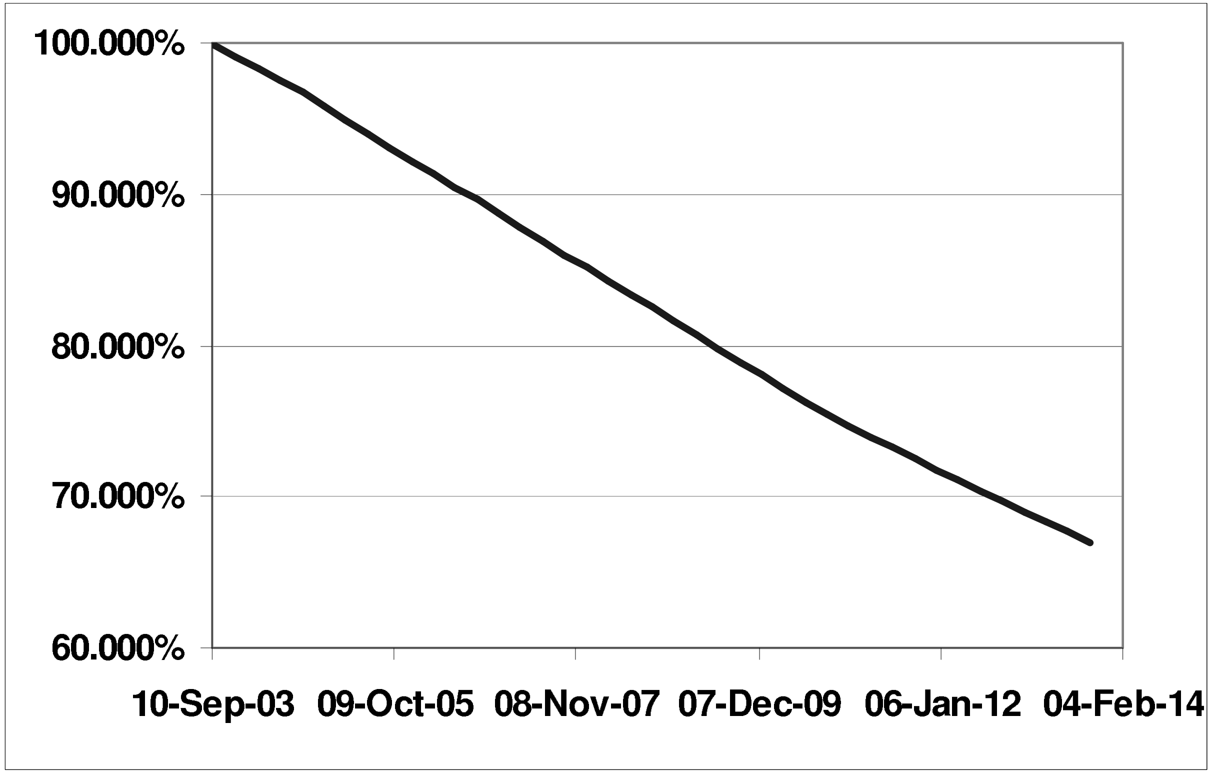


Figure 7: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on September 10th, 2003.

# November 28th, 2003

Recovery Rate = 40%

| Maturity $T_b$ (yr) | Maturity (dates) | $R_{0,b}$ |
|---|---|---|
| 1 | 20-Dec-04 | 725 |
| 3 | 20-Dec-06 | 630 |
| 5 | 20-Dec-08 | 570 |
| 7 | 20-Dec-10 | 570 |
| 10 | 20-Dec-13 | 570 |

Table 5: Maturity Dates and corresponding CDS quotes in bps relative to November 28th, 2003.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 28-Nov-03 | 12.047% | 100.000% |
| 20-Dec-04 | 12.047% | 87.824% |
| 20-Dec-06 | 6.545% | 72.736% |
| 22-Dec-08 | 8.226% | 62.581% |
| 20-Dec-10 | 10.779% | 51.640% |
| 20-Dec-13 | 7.880% | 38.872% |

Table 6: Calibration with piecewise linear intensity on November 28th, 2003.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 28-Nov-03 | 12.047% | 100.000% |
| 20-Dec-04 | 12.047% | 87.824% |
| 20-Dec-06 | 9.426% | 72.545% |
| 22-Dec-08 | 7.331% | 62.486% |
| 20-Dec-10 | 9.441% | 51.626% |
| 20-Dec-13 | 9.437% | 38.734% |

Table 7: Calibration with piecewise constant intensity on November 28th, 2003.

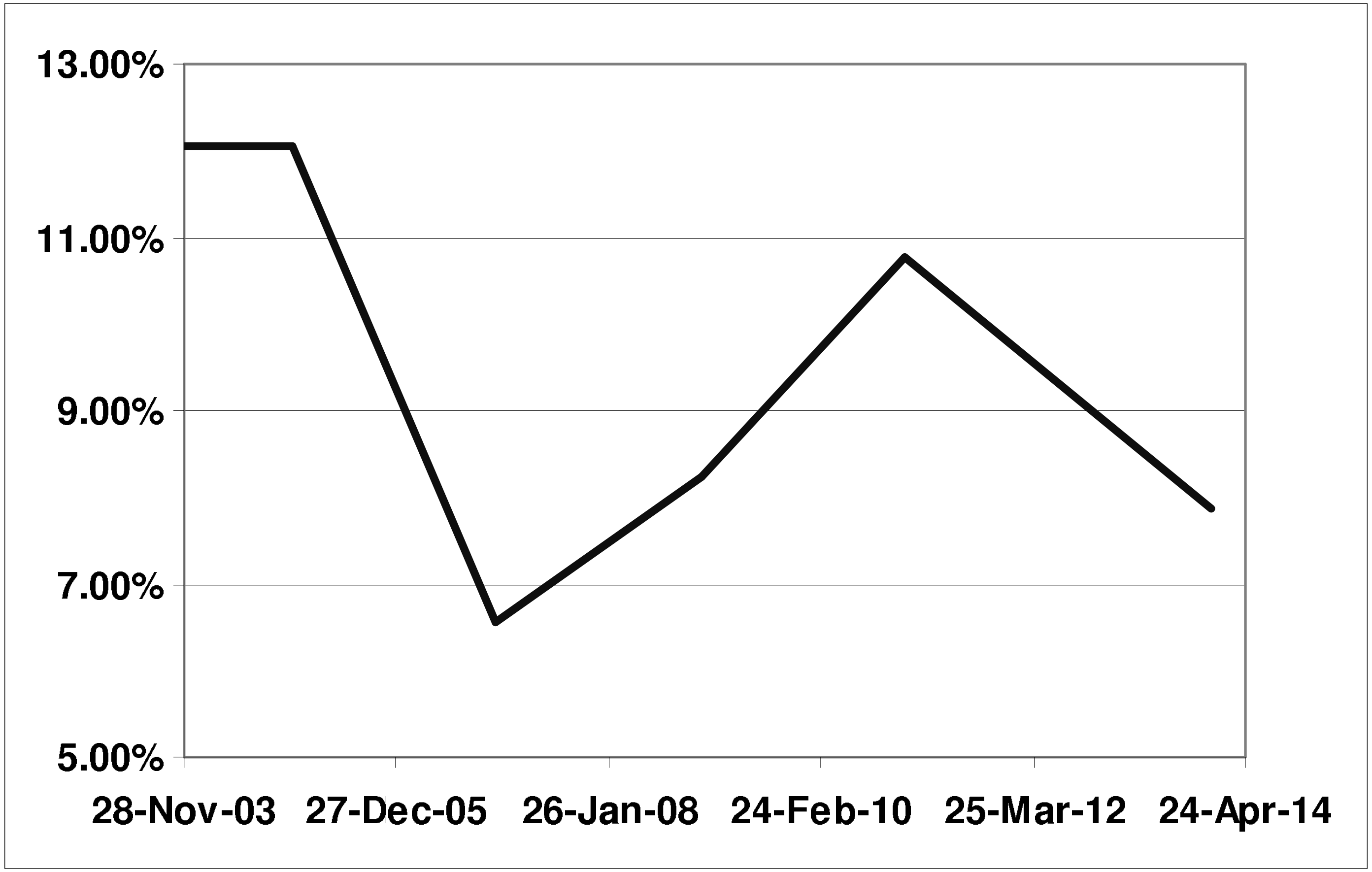


Figure 8: Piecewise linear intensity $\gamma$ calibrated on CDS quotes on November 28th, 2003.

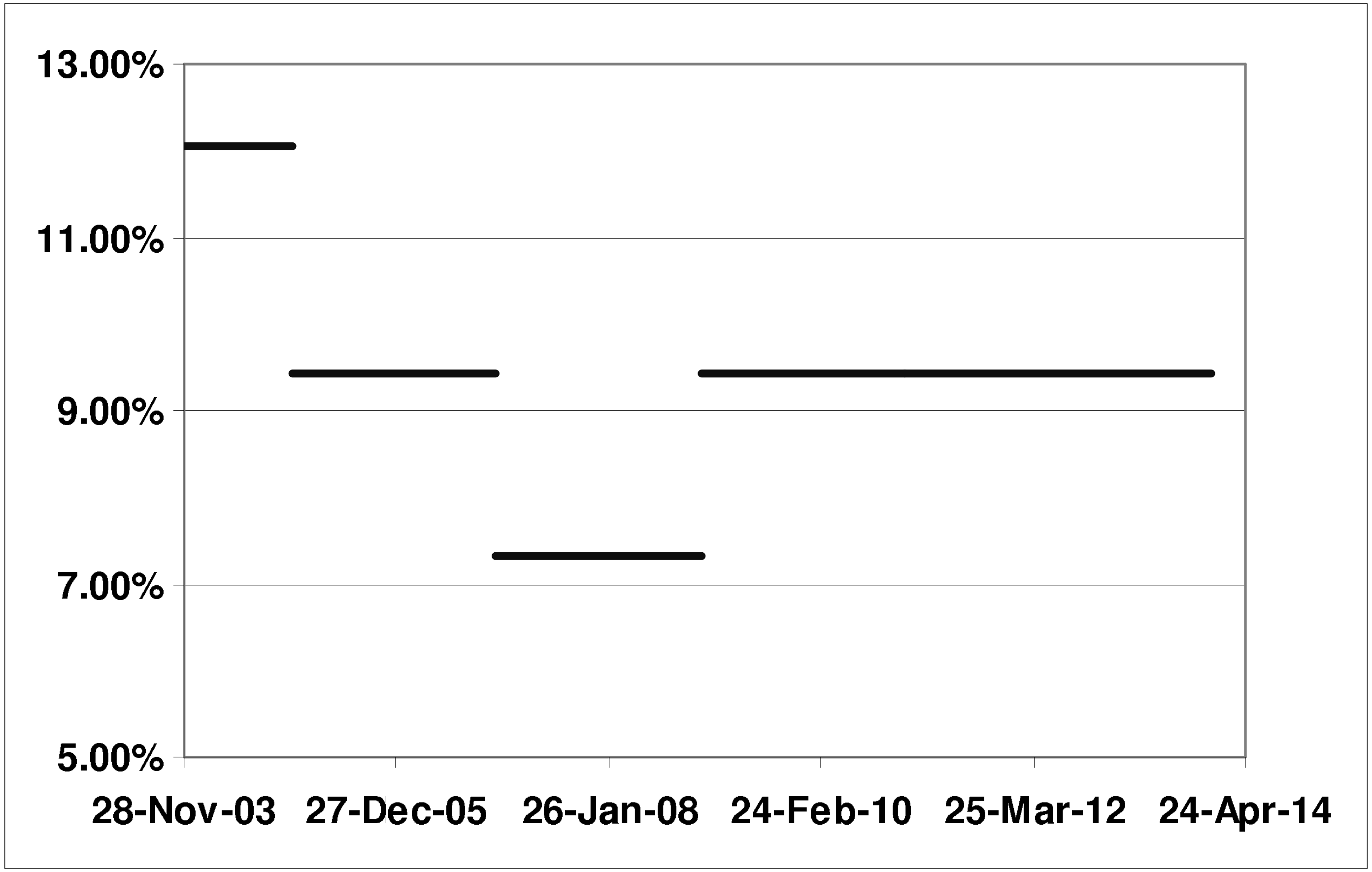


Figure 9: Piecewise constant intensity $\gamma$ calibrated on CDS quotes on November 28th, 2003.

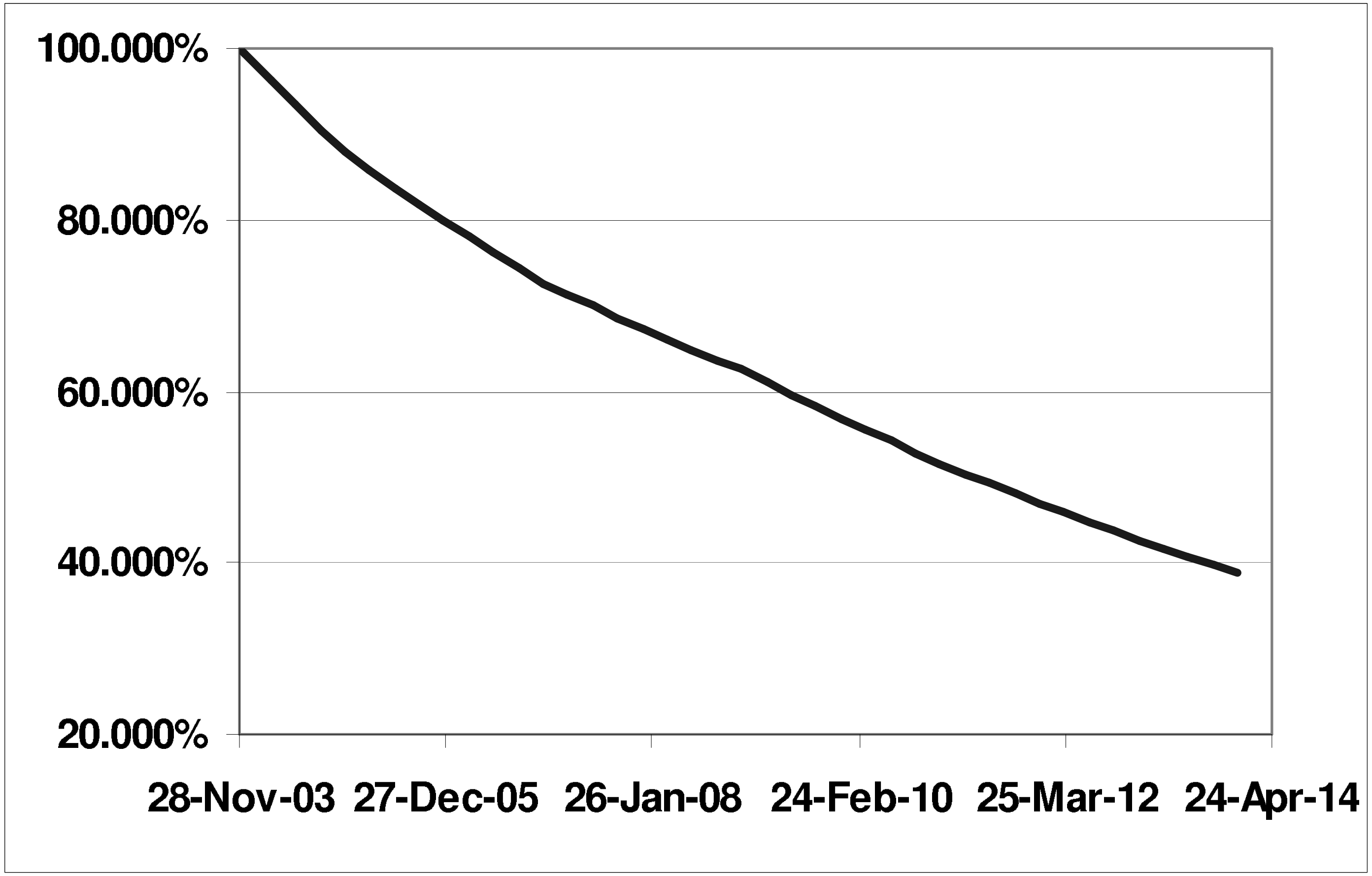


Figure 10: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on November 28th, 2003.

# December 8th, 2003

Recovery Rate = 25%

| Maturity $T_b$ (yr) | Maturity (dates) | $R_{0,b}$ |
|---|---|---|
| 1 | 20-Dec-04 | 1450 |
| 3 | 20-Dec-06 | 1200 |
| 5 | 20-Dec-08 | 940 |
| 7 | 20-Dec-10 | 850 |
| 10 | 20-Dec-13 | 850 |

Table 8: Maturity Dates and corresponding CDS quotes in bps relative to December 8th, 2003.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 08-Dec-03 | 19.272% | 100.000% |
| 20-Dec-04 | 19.272% | 81.680% |
| 20-Dec-06 | 7.263% | 62.413% |
| 22-Dec-08 | 2.393% | 56.570% |
| 20-Dec-10 | 11.205% | 49.303% |
| 20-Dec-13 | 11.318% | 34.993% |

Table 9: Calibration with piecewise linear intensity on December 8th, 2003.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 08-Dec-03 | 19.272% | 100.000% |
| 20-Dec-04 | 19.272% | 81.680% |
| 20-Dec-06 | 13.650% | 61.931% |
| 22-Dec-08 | 4.834% | 56.126% |
| 20-Dec-10 | 6.500% | 49.213% |
| 20-Dec-13 | 11.256% | 34.934% |

Table 10: Calibration with piecewise constant intensity on December 8th, 2003.

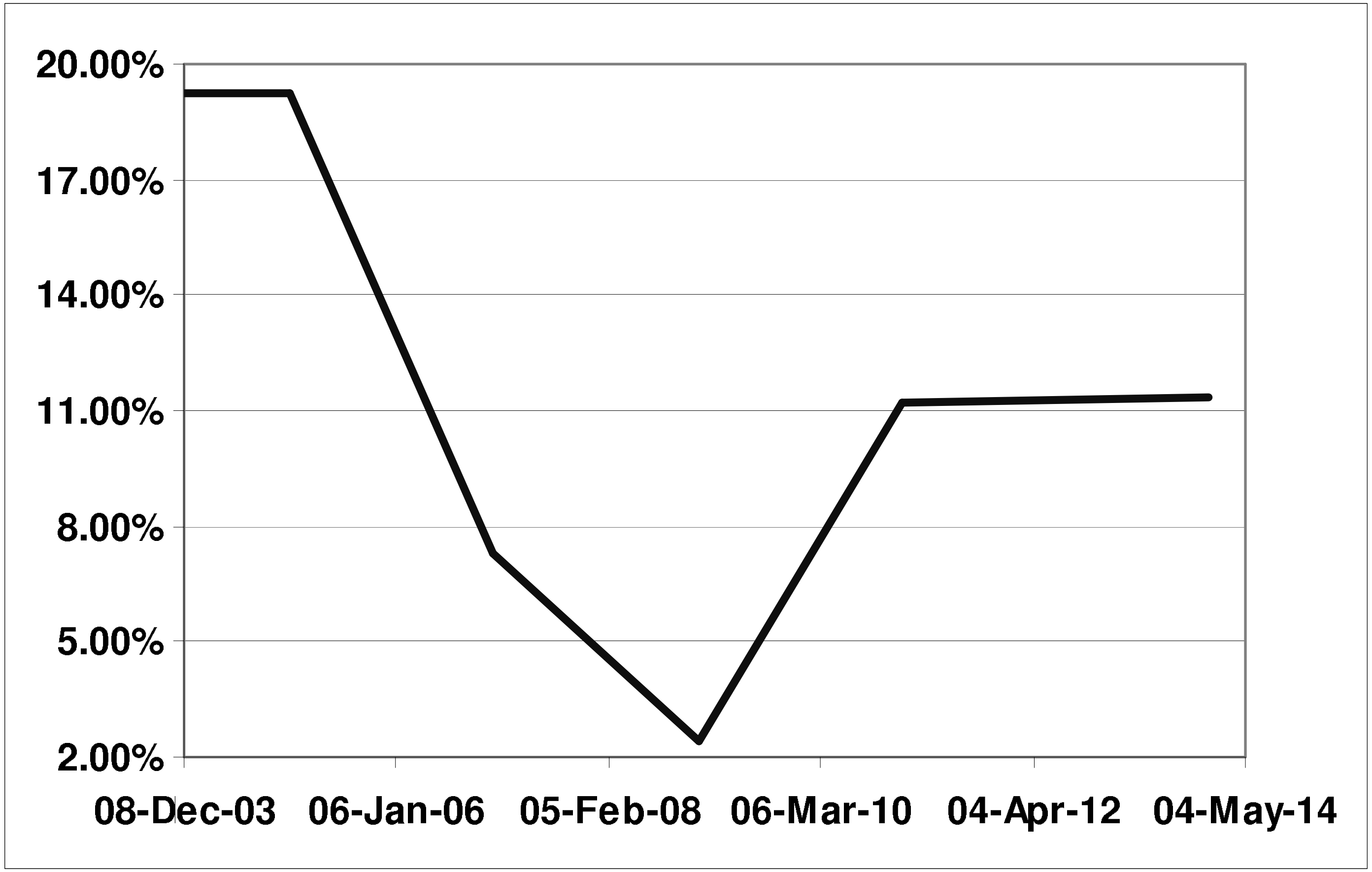


Figure 11: Piecewise linear intensity $\gamma$ calibrated on CDS quotes on December 8th, 2003.

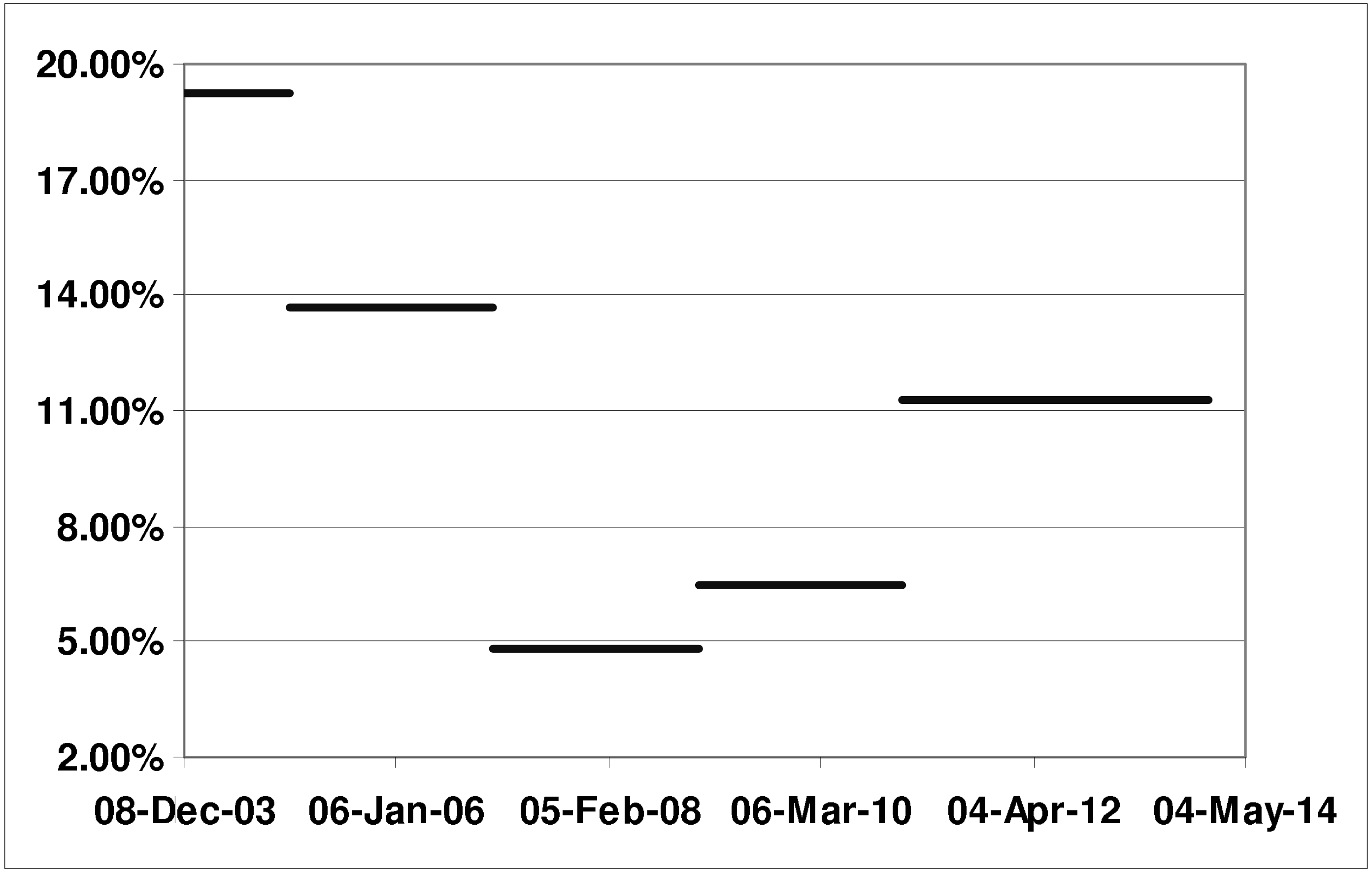


Figure 12: Piecewise constant intensity $\gamma$ calibrated on CDS quotes on December 8th, 2003.

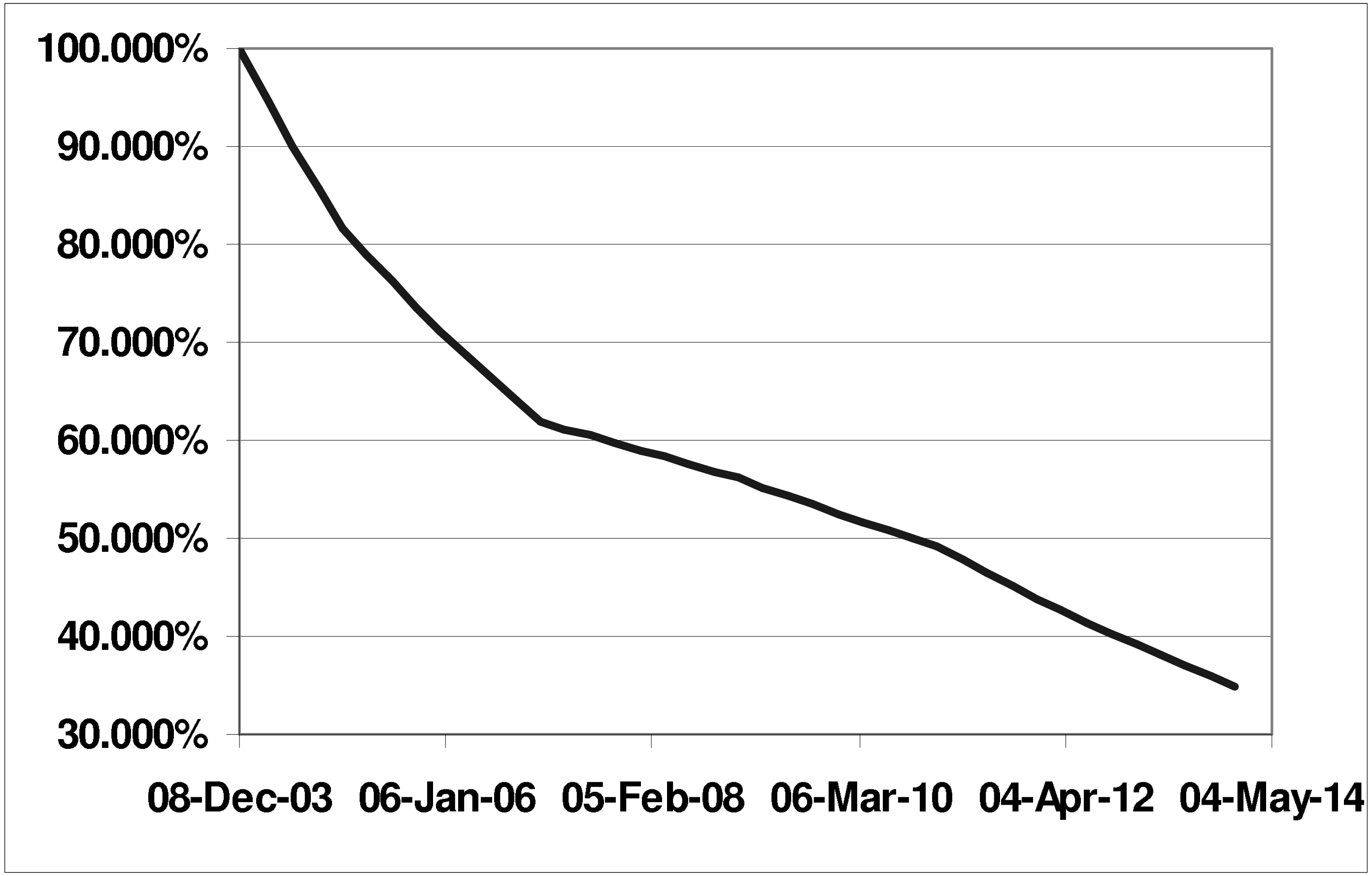


Figure 13: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on December 8th, 2003.

# December 10th, 2003

Recovery Rate = 15%

| Maturity $T_b$ (yr) | Maturity (dates) | $R_{0,b}$ |
|---|---|---|
| 1 | 20-Dec-04 | 5050 |
| 3 | 20-Dec-06 | 2100 |
| 5 | 20-Dec-08 | 1500 |
| 7 | 20-Dec-10 | 1250 |
| 10 | 20-Dec-13 | 1100 |

Table 11: Maturity Dates and corresponding CDS quotes in bps relative to December 10th, 2003.

Payoff Type: Posponed 1

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 10-Dec-03 | 55.483% | 100.000% |
| 20-Dec-04 | 55.483% | 56.018% |
| 20-Dec-06 | -61.665% | 59.642% |
| 22-Dec-08 | 84.397% | 47.321% |
| 20-Dec-10 | -86.408% | 48.293% |
| 20-Dec-13 | 123.208% | 27.581% |

Table 12: Calibration with pwise linear intensity and postponed payoff 1 on Dec 10, 2003.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 10-Dec-03 | 55.483% | 100.000% |
| 20-Dec-04 | 55.483% | 56.018% |
| 20-Dec-06 | 0.807% | 55.109% |
| 22-Dec-08 | 4.017% | 50.780% |
| 20-Dec-10 | 4.292% | 46.559% |
| 20-Dec-13 | 5.980% | 38.809% |

Table 13: Calibration with pwise const intensity and postponed payoff 1 on Dec 10, 2003.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 10-Dec-03 | 64.188% | 100.000% |
| 20-Dec-04 | 64.188% | 51.150% |
| 20-Dec-06 | -80.163% | 60.144% |
| 22-Dec-08 | 108.007% | 45.299% |
| 20-Dec-10 | -113.620% | 47.944% |
| 20-Dec-13 | 162.645% | 22.732% |

Table 14: Calibration with pwise linear intensity and postponed payoff 2.

| date | intensity $\gamma$ | survival pr $\exp(-\Gamma)$ |
|---|---|---|
| 10-Dec-03 | 64.188% | 100.000% |
| 20-Dec-04 | 64.188% | 51.150% |
| 20-Dec-06 | -3.270% | 54.657% |
| 22-Dec-08 | 3.900% | 50.484% |
| 20-Dec-10 | 4.282% | 46.297% |
| 20-Dec-13 | 6.065% | 38.491% |

Table 15: Calibration with piecewise constant intensity and postponed payoff of kind 2.

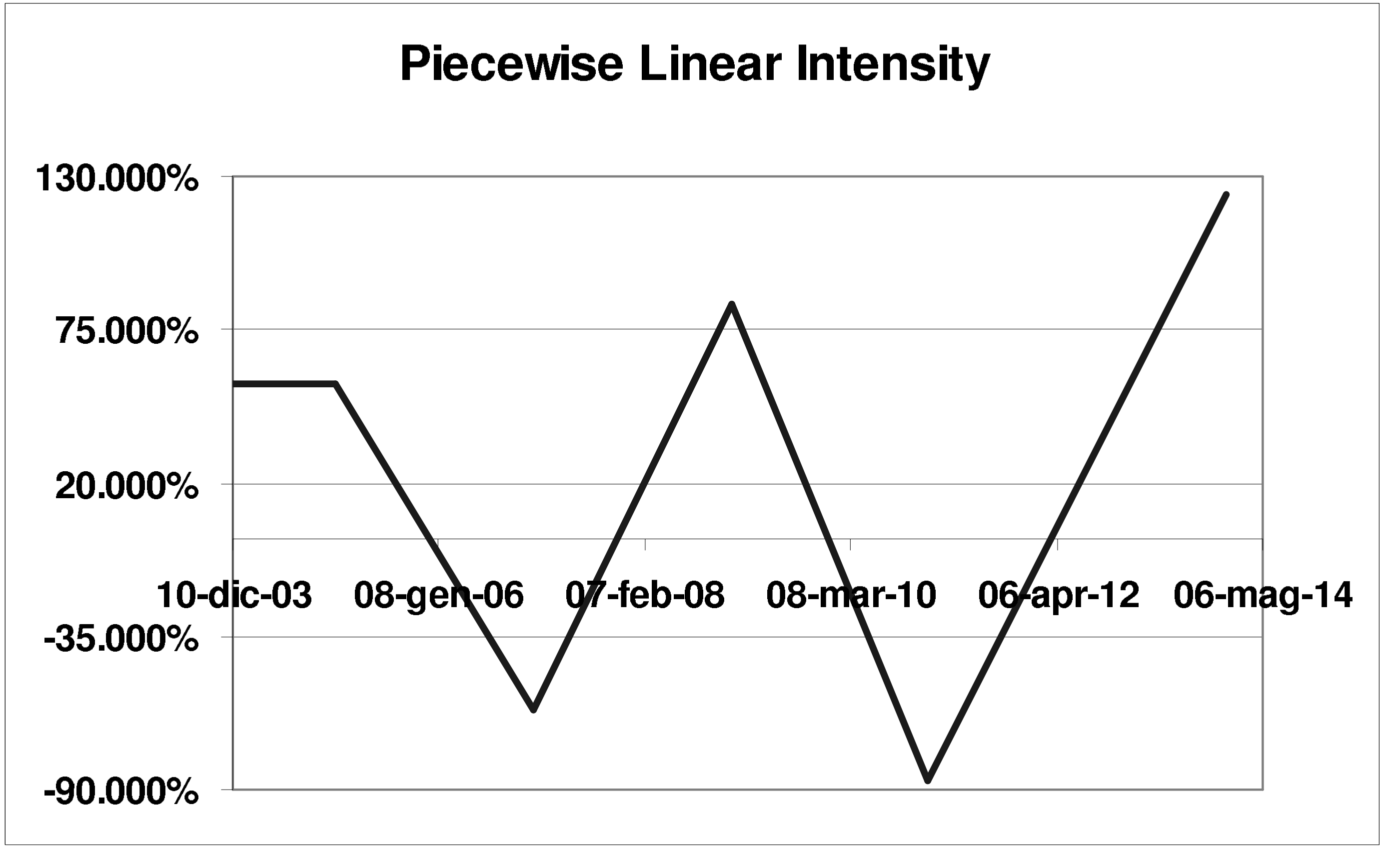


Figure 14: Piecewise linear intensity $\gamma$ calibrated on CDS quotes on December 10th, 2003, with postponed payoff of kind 1.

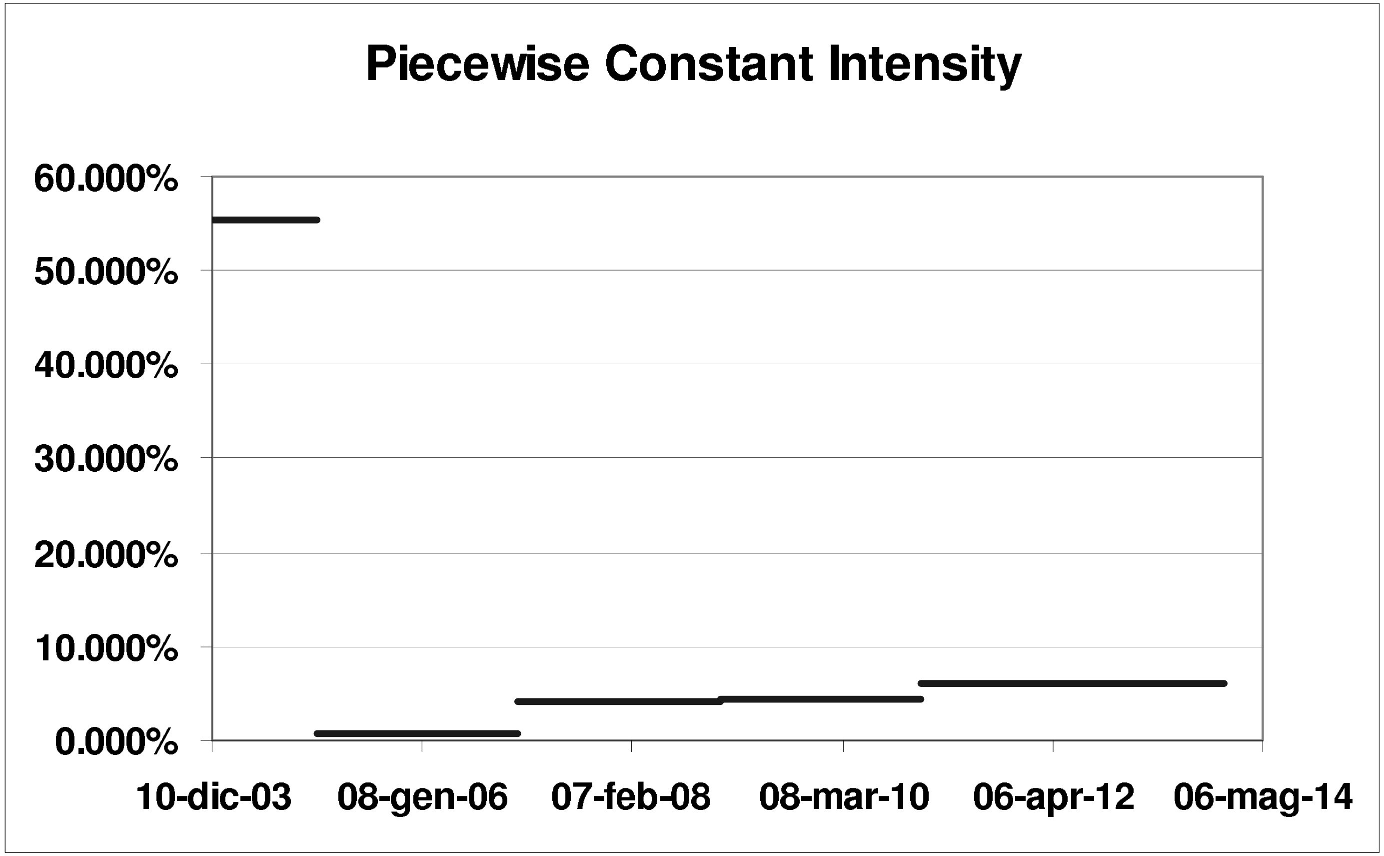


Figure 15: Piecewise constant intensity $\gamma$ calibrated on CDS quotes on December 10th, 2003, with postponed payoff of kind 1.

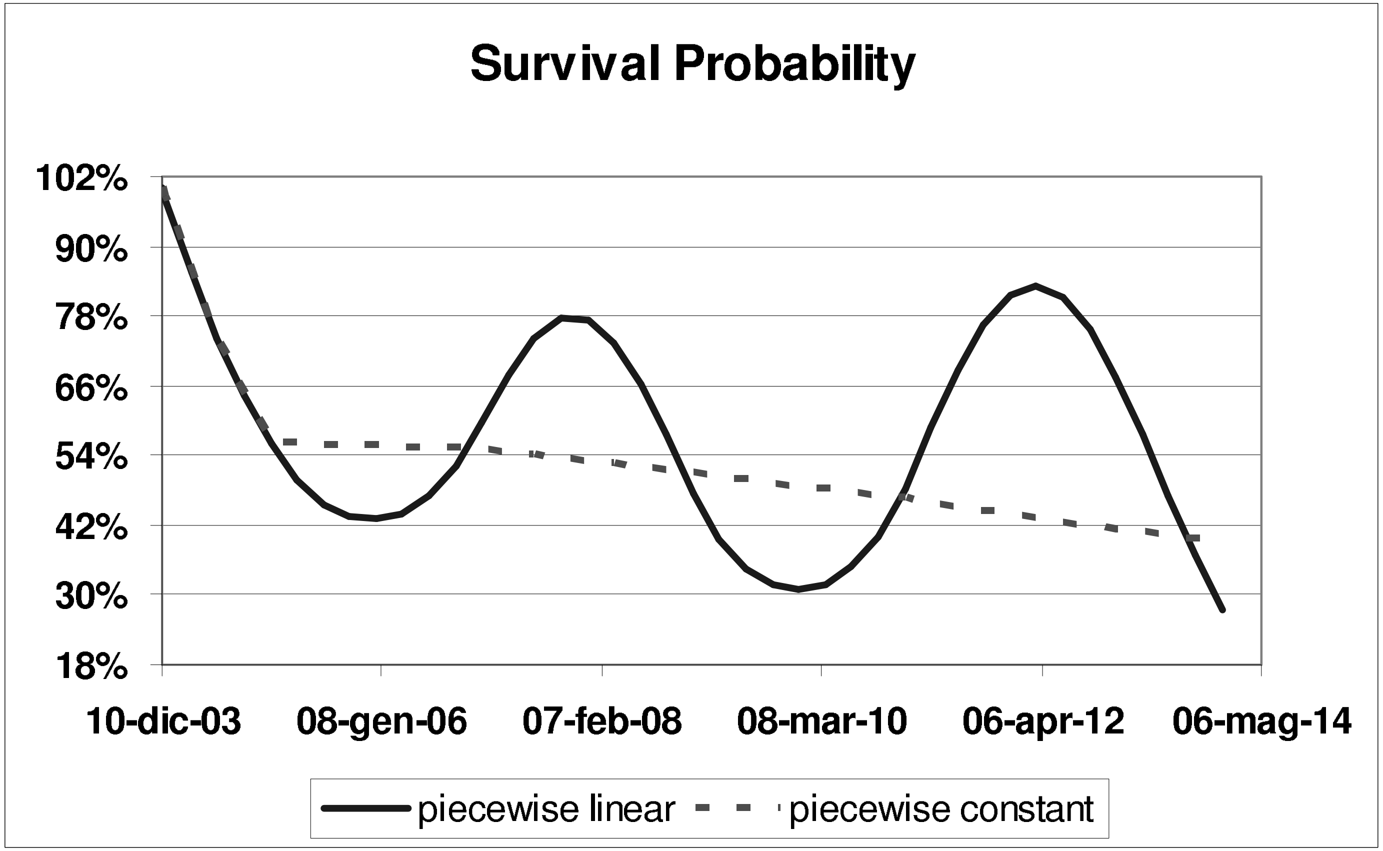


Figure 16: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on December 10th, 2003, with postponed payoff of kind 1.

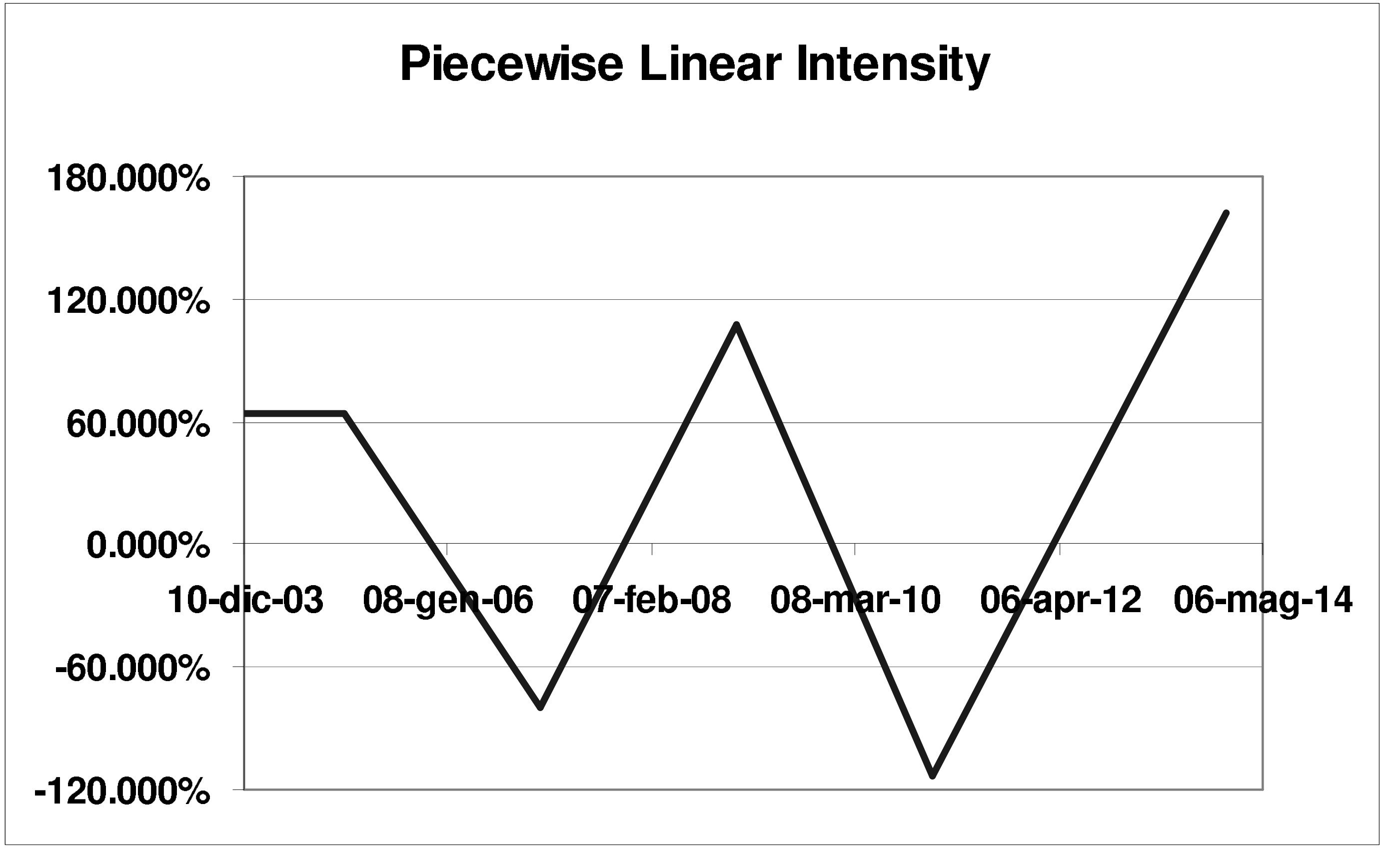


Figure 17: Piecewise linear intensity $\gamma$ calibrated on CDS quotes on December 10th, 2003, with postponed payoff of kind 2.

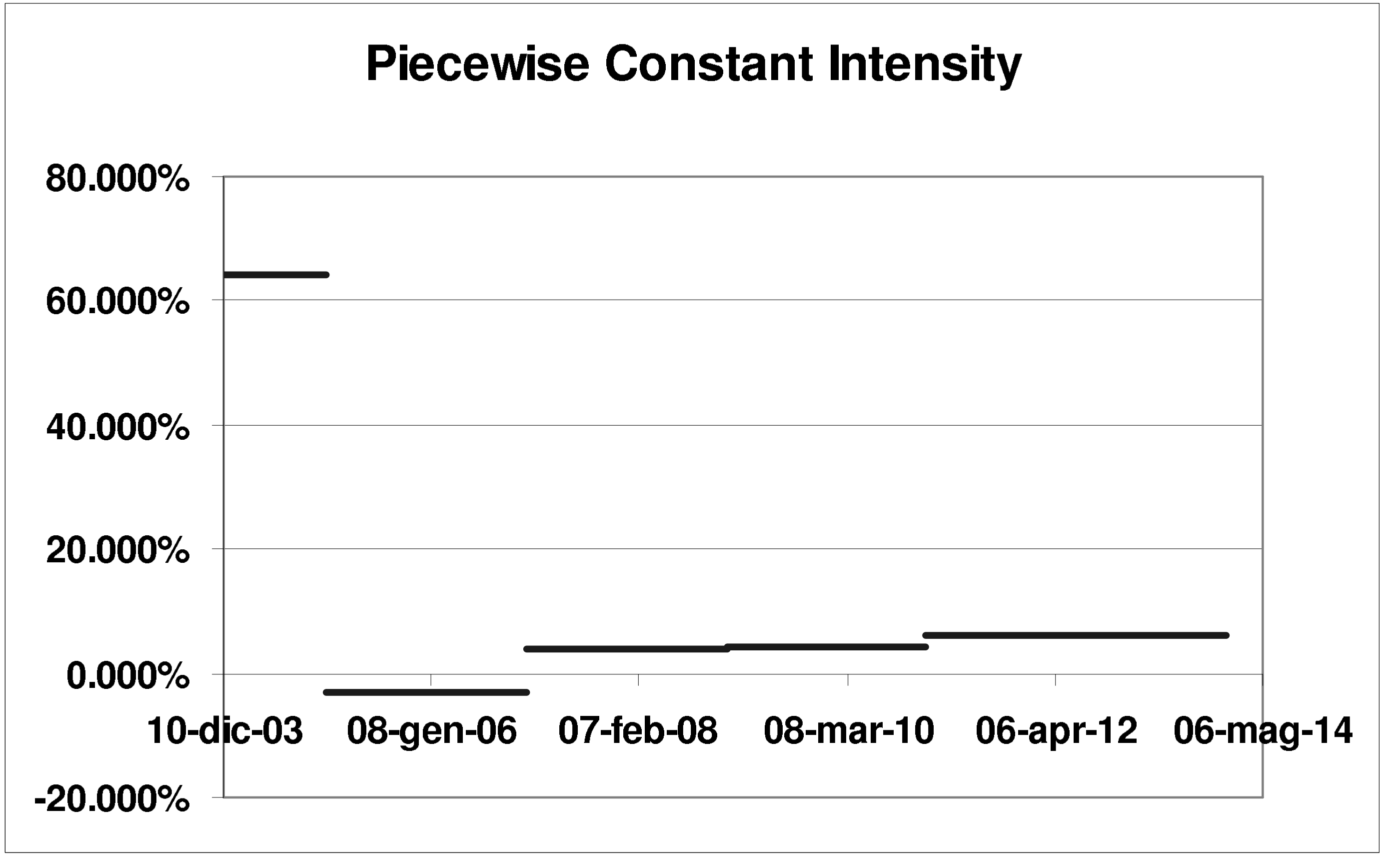


Figure 18: Piecewise constant intensity $\gamma$ calibrated on CDS quotes on December 10th, 2003, with postponed payoff of kind 2.

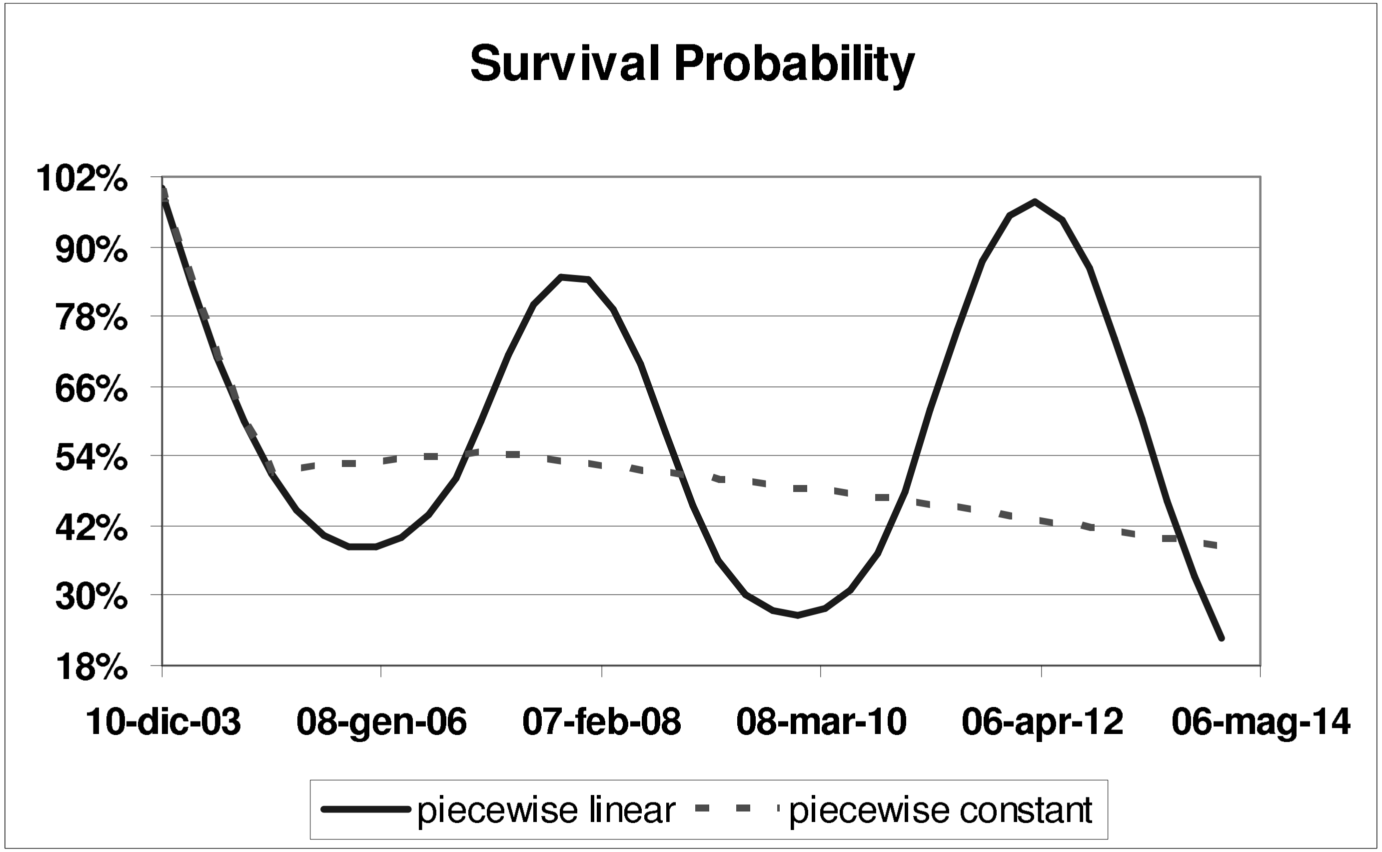


Figure 19: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on December 10th, 2003, with postponed payoff of kind 2.

# Two Important Technicalities

**i): Dependence default/ interest rates**. Consider interest rates to be stochastic and to be driven by Brownian Motions as sources of randomness. This happens in most interest rate models such as Hull White, CIR, Black Derman Toy, Black Karasinski, or the Libor market model, or the Heath Jarrow Morton framework.

Since a Poisson process and a Brownian motion defined on the same space are independent, *we can assume the stochastic discount* $D(s,t) = \exp(-\int_s^t r_u du)$ *(Brownian motions) and the default time* $\tau$ *(Poisson process) to be* **independent** *under deterministic intensities* for $\tau$

# Two Important Technicalities

This is implicit in what we have seen before: if

$$\tau = \Gamma^{-1}(\xi)$$

with $\Gamma$ deterministic Hazard function and $\xi$ exponential RV **independent of anything else** (and of interest rates in particular), clearly $\tau$ will be **independent of interest rates**.

This is useful: For example, consider the bond price. By independence

$$\mathbb{E}[D(0,T)1_{\{\tau>T\}}] = \mathbb{E}[D(0,T)]\mathbb{E}[1_{\{\tau>T\}}] = P(0,T)\mathbb{Q}(\tau > T)$$

**defaultable zero-coupon bond = riskless zero-coupon bond * survival probability**

Our only hope for correlating default and interest rate is to make THE INTENSITY STOCHASTIC $\Gamma \to \Lambda$ AND TO CORRELATE IT TO INTEREST RATES:

$$\tau = \Lambda^{-1}(\xi), \quad \Lambda(t) = \int_0^t \lambda_s ds, \quad \text{corr}(d\lambda, dr) = \rho$$

# Two Important Technicalities

**ii) The filtration switching formula: Computing $\mathcal{G}$ expectations via $\mathcal{F}$ ones**.

Recall: $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$, $\mathcal{F}_t$= "info-on-default-free-markets-up-to-$t$"; $\sigma(\{\tau < u\}, u \leq t) =$ "info if default occurred before $t$, and, if so, when exactly".

Credit derivatives prices have to be computed through risk neutral expectation given the information in $\mathcal{G}$: interest rates, FX rates, credit spreads ..., AND default monitoring.

$$\text{Price} = \mathbb{E}(1_{\{\tau > T\}}\text{Payoff}|\mathcal{G}_t)$$

However, we would like to be able to price given only the information in $\mathcal{F}$: interest rates, FX rates, credit spreads..., WITHOUT default monitoring.

# Two Important Technicalities

In a Cox process setting (assuming $\mathcal{F}_t$ progressively measurable and positive stochastic intensity $\lambda$ with integrable paths, under very general measurability conditions for the payoff, typically the payoff is assumed to be $\mathcal{G}_\infty$-measurable, i.e. known at time infinity) and for $t < T$ we have

$$\mathbb{E}(1_{\{\tau>T\}}\text{Payoff}|\mathcal{G}_t) = \frac{1_{\{\tau>t\}}}{\mathbb{Q}\{\tau > t|\mathcal{F}_t\}}\mathbb{E}(1_{\{\tau>T\}}\text{Payoff}|\mathcal{F}_t)$$

**Switching from $\mathcal{G}$ expectations to $\mathcal{F}$ expectations is important because for some variables the $\mathcal{F}$ conditional expectations are easier to compute. Also, this helps to avoid problems on non-equivalence of probability measures when defining market models for CDS Options**

# Default Simulation in reduced form models

Assume we have obtained the (positive) $\gamma$'s from CDS market quotes.

How do we simulate the default time $\tau$? Recall that $\tau = \Gamma^{-1}(\xi)$. Then

a) Simulate a standard exponential RV $\xi$. This can be done for example by simulating samples $u_1, \ldots, u_m$ from a uniform variable $U$ in $[0, 1]$ and then taking $\xi_1 := -\ln(1 - u_1), \ldots, \xi_m = -\ln(1 - u_m)$.

b) For each $\xi_j$, solve $\Gamma(\tau_j) = \xi_j$ in $\tau_j$; the obtained solutions are the simulated default times $\tau_1, \ldots, \tau_m$.

Possible problem: if some obtained $\xi$'s are very large, we may not have $\Gamma$ going far enough in time to solve the equation, because our CDS data and Bond data from which we deduced $\Gamma$ stopped before.

# Default Simulation in reduced form models

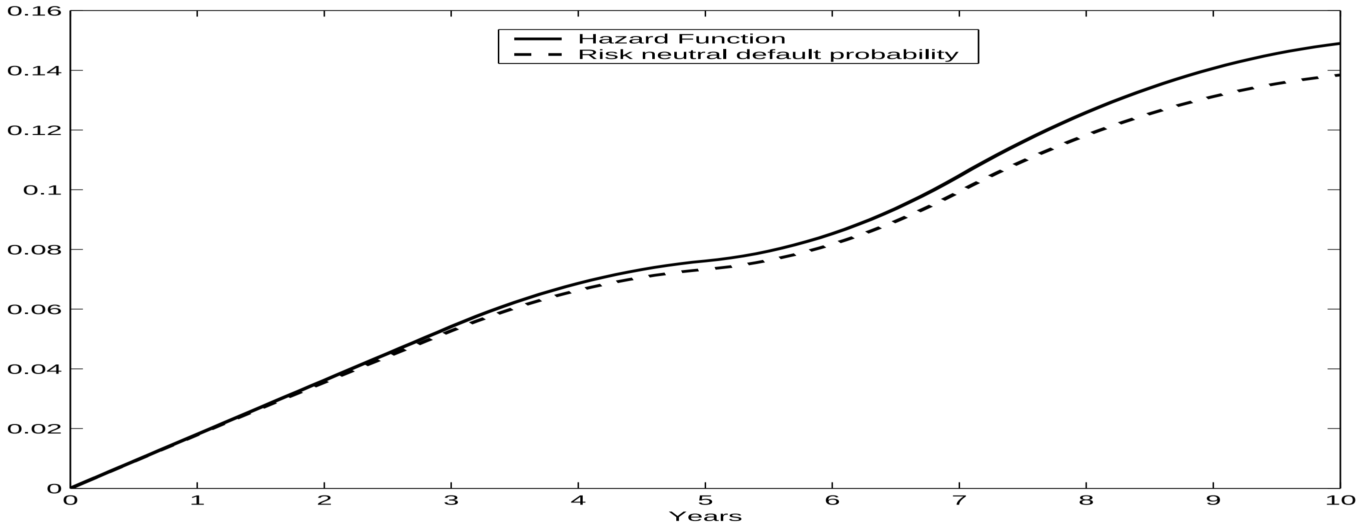


Figure 20: Hazard function $\Gamma$ and risk-neutral default probability for MLynch CDS's, October 25, 2002

If in this case we have obtained a $\xi_j = 0.3$, we cannot solve $\Gamma(\tau_j) = 0.3$ from our data, unless we extend in some way the graph of $\Gamma$ beyond its maturity.
Of course the probability of $\xi$ being larger than 0.14, our final $\Gamma$, is large: $\mathbb{Q}(\xi > 0.14) = \exp(-0.14) = 0.87$, so that this is an important problem we can find very often from the simulated $\xi_i$ (in our example in $87\%$ of simulations)

# Default Simulation in reduced form models

A second problem of the simulation is that in presence of default, to obtain an acceptable precision with a Monte-Carlo algorithm, it is unfortunately necessary to simulate a quite large number of scenarios.

Indeed, variances are quite large in relative terms, due essentially to indicator terms in the payoffs, such as for example $\mathbf{1}_{\{\tau<T\}}\text{L}_{\text{GD}}D(0,\tau)$. A quick example can help us to clarify this important second point.

Assume we want a MC simulation for $\mathbb{E}\mathbf{1}_{\{\tau<T\}}$. Compute the variance

$$\text{Var}(\mathbf{1}_{\{\tau<T\}}) = E\mathbf{1}^2_{\{\tau<T\}} - (E\mathbf{1}_{\{\tau<T\}})^2 = E\mathbf{1}_{\{\tau<T\}} - (E\mathbf{1}_{\{\tau<T\}})^2.$$

Consider for example the ML data given in the Figure above and take $T = 5y$. Notice that $E\mathbf{1}_{\{\tau<T\}}$ is the risk neutral probability to default in 5y for ML. From the graph we see that this is about 0.07. Then the above variance is about $0.07 - 0.07^2 = 0.0651$, and the standard deviation is $\sqrt{0.0651} = 0.2551$.

# Default Simulation in reduced form models

We know that the standard error in the Monte Carlo method is given by the standard deviation of the object we are simulating divided by the square root of the number of paths. So we have that the standard error is about $0.2551/\sqrt{\text{npaths}}$. Now, we are estimating a quantity that is about $0.07$ and we would like to have a standard error below one basis point. But if we wish our standard error to be below one basis point (i.e. 1/10000) we need to set

$$0.2551/\sqrt{\text{npaths}} < 1/10000 \;\Rightarrow \text{npaths} > (10000 * 0.2551)^2 = 6507601.$$

# Default Simulation in reduced form models

We may slightly improve the situation by setting a threshold barrier $\bar{B}$ such that we will be interested only in $\Gamma$'s with $\Gamma(T) < \bar{B}$. This means we will be able to retrieve $\tau$'s with $\Gamma(\tau) < \bar{B}$, i.e. $\tau < \Gamma^{-1}(\bar{B})$. Default times larger than this will not be simulated.

This is natural since most payoffs become known when the default time exceeds a given final maturity $T_b$, so that there is no need to simulate them.

Indeed, if $\Pi$ is a payoff we can write the price as

$$\mathbb{E}\ \Pi = \mathbb{E}[\Pi|\Gamma(\tau) < \bar{B}]\mathbb{Q}(\Gamma(\tau) < \bar{B}) + \mathbb{E}[\Pi|\Gamma(\tau) > \bar{B}]\mathbb{Q}(\Gamma(\tau) > \bar{B})$$

$$= \mathbb{E}[\Pi|\xi < \bar{B}]\mathbb{Q}(\xi < \bar{B}) + \mathbb{E}[\Pi|\xi > \bar{B}]\mathbb{Q}(\xi > \bar{B})$$
$$= \mathbb{E}[\Pi|\Gamma(\tau) < \bar{B}](1 - e^{-\bar{B}}) + \mathbb{E}[\Pi|\Gamma(\tau) \geq \bar{B}]e^{-\bar{B}}.$$

Now, in payoffs that become trivial for large enough default times (like Bonds or CDS for example) the second expected value on the right hand side need not be computed.

# Default Simulation in reduced form models

$$\mathbb{E}\Pi = \mathbb{E}[\Pi|\xi < \bar{B}](1 - e^{-\bar{B}}) + \mathbb{E}[\Pi|\xi \geq \bar{B}]e^{-\bar{B}}.$$

Since the second term is usually known without simulation, the idea is then to simulate default times $\tau = \Gamma^{-1}(\xi)$ **conditional on** $\xi := \Gamma(\tau) < \bar{B}$.

So, if $\xi$ is an exponential random variable with parameter one we just simulate $\xi|\xi < \bar{B}$, whose density is easily seen to be

$$p_{\xi|\xi<\bar{B}}(u) = 1_{\{u<\bar{B}\}}e^{-u}/(1 - e^{-\bar{B}}).$$

From the exponential distribution we see that simulating $N$ scenarios for $\xi$ amounts to simulate $N(1 - e^{-\bar{B}})$ scenarios with $\xi < \bar{B}$ and $Ne^{-\bar{B}}$ with $\xi \geq \bar{B}$. So in turn simulating $M = N(1 - e^{-\bar{B}})$ scenarios for $\xi < \bar{B}$, as we will do, amounts to simulate in total $N = M/(1 - e^{-\bar{B}})$ scenarios, the extra scenarios corresponding to the known value of the payoff when $\tau > \Gamma^{-1}(\bar{B})$.

# Default Simulation in reduced form models

Simulating $M$ scenarios for $\xi < \bar{B} \Rightarrow$ simulating $M/(1 - e^{-\bar{B}})$ total scenarios

Dividing by $1 - e^{-\bar{B}}$ may help us increase efficiency (in our examples typically it increases the number of scenarios by a factor 10), but a large amount of scenarios remains to be generated, and the time needed for Monte Carlo simulation remains large.

For example, in the ML example above, if the payoff becomes trivial after 10y, we may consider only $\tau < 10$, i.e. $\Gamma(\tau) < \Gamma(10) = 0.15 =: \bar{B}$. Then $1/(1 - e^{-\bar{B}}) = 7.18$.

If the payoff "stops" after 5y, since $\Gamma(5y) = 0.08$, we have $1/(1 - e^{-\bar{B}}) = 13$.

In presence of much larger default probabilities, like for Parmalat on december 8 2003, where we had a $35\%$ survival probability in 10y, i.e. $\Gamma(10y) = 1.0518 =: \bar{B}$, we would obtain a much poorer reduction: $1/(1 - e^{-\bar{B}}) = 1.54$.

# Default Simulation in reduced form models

In all cases the above trick can be combined with a Control Variate technique to reduce the variance of the Monte Carlo estimate (and thus the standard error).

First some notation.

We consider the multi-name situation directly.

Suppose we have to value a financial payoff depending on the default times of $n$ names. Let the discounted payoff be

$$\Pi := X(\tau_1, \ldots, \tau_n)$$

i.e. a given function of the default times of each name.

Assume we generate $n_p$ scenarios for the default times, each scenario denoted by upper index: $\tau_i^j$ denotes the $j$-th scenario of the default time for name $i$.
**Let $\Pi^j$ be the corresponding $j$-th scenario for the discounted payoff.**

# Default Simulation in reduced form models

The Monte Carlo price of our payoff is computed, based on the simulated paths, as

$$E[\Pi(T)] = \sum_{j=1}^{n_p} \Pi^j / n_p$$

where the default times $\tau$ in $\Pi$ have been simulated under the risk neutral measure.

We wish to have an estimate of the error we have when estimating the true expectation $E(\Pi)$ by its Monte Carlo estimate $\sum_{j=1}^{n_p} \Pi^j / n_p$. To do so, the classic reasoning is as follows.

Let us view $(\Pi^j)_j$ as a sequence of independent identically distributed (iid) random variables, distributed as $\Pi$. By the central limit theorem, we know that under suitable assumptions one has

$$\frac{\sum_{j=1}^{n_p}(\Pi^j - E(\Pi))}{\sqrt{n_p}\,\text{Std}(\Pi)} \to \mathcal{N}(0,1),$$

in law, as $n_p \to \infty$

# Default Simulation in reduced form models

$$\frac{\sum_{j=1}^{n_p}(\Pi^j - E(\Pi))}{\sqrt{n_p}\,\mathsf{Std}(\Pi)} \to \mathcal{N}(0,1),$$

in law, as $n_p \to \infty$, from which we may write, approximately and for large $n_p$:

$$\frac{\sum_{j=1}^{n_p}\Pi^j}{n_p} - E(\Pi) \sim \frac{\mathsf{Std}(\Pi)}{\sqrt{n_p}}\,\mathcal{N}(0,1).$$

It follows that

$$\mathbb{Q}\left\{\left|\frac{\sum_{j=1}^{n_p}\Pi^j}{n_p} - E(\Pi)\right| < \epsilon\right\} = \mathbb{Q}\left\{|\mathcal{N}(0,1)| < \epsilon\,\frac{\sqrt{n_p}}{\mathsf{Std}(\Pi)}\right\} = 2\Phi\left(\epsilon\,\frac{\sqrt{n_p}}{\mathsf{Std}(\Pi)}\right) - 1,$$

where as usual $\Phi$ denotes the cumulative distribution function of the standard Gaussian random variable.

# Monte Carlo pricing : standard error

$$\mathbb{Q}\left\{ \left| \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} - E(\Pi) \right| < \epsilon \right\} = 2\Phi\left( \epsilon \, \frac{\sqrt{n_p}}{\mathrm{Std}(\Pi)} \right) - 1,$$

The above equation gives the probability that our Monte Carlo estimate $\sum_{j=1}^{n_p} \Pi^j / n_p$ is not farther than $\epsilon$ from the true expectation $E(\Pi)$ we wish to estimate. Typically, one sets a desired value for this probability, say 0.98, and derives $\epsilon$ by solving

$$2\Phi\left( \epsilon \, \frac{\sqrt{n_p}}{\mathrm{Std}(\Pi)} \right) - 1 = 0.98.$$

For example, since we know from the $\Phi$ tables that

$$2\Phi(z) - 1 = 0.98 \iff \Phi(z) = 0.99 \iff z \approx 2.33,$$

we have that

$$\epsilon = 2.33 \, \frac{\mathrm{Std}(\Pi)}{\sqrt{n_p}}.$$

# Monte Carlo pricing : standard error

The true value of $E(\Pi)$ is thus inside the "window"

$$\left[\frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} - 2.33\,\frac{\mathsf{Std}(\Pi)}{\sqrt{n_p}}, \quad \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + 2.33\,\frac{\mathsf{Std}(\Pi)}{\sqrt{n_p}}\right]$$

with a $98\%$ probability. This is called a $98\%$ confidence interval for $E(\Pi)$. Other typical confidence levels are given in Table 16.

| $2\Phi(z) - 1$ | $z \approx$ |
|---|---|
| 99% | 2.58 |
| 98% | 2.33 |
| 95.45% | 2 |
| 95% | 1.96 |
| 90% | 1.65 |
| 68.27% | 1 |

Table 16: Confidence levels

# Monte Carlo pricing : standard error

$$\left[\frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} - 2.33\,\frac{\mathsf{Std}(\Pi)}{\sqrt{n_p}},\quad \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + 2.33\,\frac{\mathsf{Std}(\Pi)}{\sqrt{n_p}}\right]$$

We can see that, all things being equal, as $n_p$ increases, the window shrinks as $1/\sqrt{n_p}$, which is worse than $1/n_p$. If we need to reduce the window size to one tenth, we have to increase the number of scenarios by a factor 100. Sometimes, to reach a chosen accuracy (a small enough window), we need to take a huge number of scenarios $n_p$. When this is too time-consuming, there are "variance-reduction" techniques that may be used to reduce the above window size.

# Monte Carlo pricing : standard error

A more fundamental problem with the above window is that the true standard deviation $\text{Std}(\Pi)$ of the payoff is usually unknown. This is typically replaced by the known sample standard deviation obtained by the simulated paths,

$$(\widehat{\text{Std}}(\Pi; n_p))^2 := \sum_{j=1}^{n_p} (\Pi^j)^2/n_p - (\sum_{j=1}^{n_p} \Pi^j/n_p)^2$$

and the actual $98\%$ Monte Carlo window we compute is

$$\left[\frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} - 2.33\,\frac{\widehat{\text{Std}}(\Pi; n_p)}{\sqrt{n_p}},\quad \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + 2.33\,\frac{\widehat{\text{Std}}(\Pi; n_p)}{\sqrt{n_p}}\right]. \tag{6}$$

To obtain a $95\%$ (narrower) window it is enough to replace 2.33 by 1.96, and to obtain a (still narrower) $90\%$ window it is enough to replace 2.33 by 1.65. All other sizes may be derived by the $\Phi$ tables.

# Monte Carlo pricing : Control Variate

We know that in some cases, to obtain a $98\%$ window whose (half-) width $2.33\,\widehat{\mathsf{Std}}(\Pi; n_p)/\sqrt{n_p}$ is small enough, we are forced to take a huge number of paths $n_p$. This can be a problem for computational time. A way to reduce the impact of this problem is, for a given $n_p$ that we deem to be large enough, to find alternatives that reduce the variance $(\widehat{\mathsf{Std}}(\Pi; n_p))^2$, thus narrowing the above window without increasing $n_p$.

One of the most effective methods to do this is the control variate technique.

We begin by selecting an alternative payoff $\Pi^{\mathsf{an}}$ which we know how to evaluate analytically, in that

$$E(\Pi^{\mathsf{an}}) = \pi^{\mathsf{an}}$$

is known. When we simulate our original payoff $\Pi$ we now simulate also the analytical payoff $\Pi^{\mathsf{an}}$ as a function of the same scenarios for the underlying variables $F$. We define a new control-variate estimator for $E\Pi$ as

$$\widehat{\Pi}_c(\gamma; n_p) := \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + \gamma \left( \frac{\sum_{j=1}^{n_p} \Pi^{\mathsf{an},j}}{n_p} - \pi^{\mathsf{an}} \right),$$

# Monte Carlo pricing : Control Variate

$$\widehat{\Pi}_c(\gamma; n_p) := \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + \gamma \left( \frac{\sum_{j=1}^{n_p} \Pi^{\mathsf{an},j}}{n_p} - \pi^{\mathsf{an}} \right),$$

with $\gamma$ a constant to be determined. When viewing $\Pi^j$ as iid copies of $\Pi$ and $\Pi^{\mathsf{an},j}$ as iid copies of $\Pi^{\mathsf{an}}$, the above estimator remains unbiased, since we are subtracting the true known mean $\pi^{\mathsf{an}}$ from the correction term in $\gamma$. So, once we have found that the estimator has not been biased by our correction, we may wonder whether our correction can be used to lower the variance.

Consider the random variable

$$\Pi_c(\gamma) := \Pi + \gamma(\Pi^{\mathsf{an}} - \pi^{\mathsf{an}})$$

whose expectation is the $E(\Pi)$ we are estimating, and compute

$$\mathsf{Var}(\Pi_c(\gamma)) = \mathsf{Var}(\Pi) + \gamma^2 \mathsf{Var}(\Pi^{\mathsf{an}}) + 2\gamma \mathsf{Corr}(\Pi, \Pi^{\mathsf{an}}) \mathsf{Std}(\Pi) \mathsf{Std}(\Pi^{\mathsf{an}}),$$

We may minimize this function of $\gamma$ by differentiating and setting the first derivative to zero.

# Monte Carlo pricing : Control Variate

We obtain easily that the variance is minimized by the following value of $\gamma$: $\gamma^* := -\text{Corr}(\Pi, \Pi^{\text{an}})\text{Std}(\Pi) \;/\text{Std}(\Pi^{\text{an}})$. By plugging $\gamma = \gamma^*$ into the above expression, we obtain easily

$$\text{Var}(\Pi_c(\gamma^*)) = \text{Var}(\Pi)(1 - \text{Corr}(\Pi, \Pi^{\text{an}})^2),$$

from which we see that $\Pi_c(\gamma^*)$ has a smaller variance than our original $\Pi$, the smaller this variance the larger (in absolute value) the correlation between $\Pi$ and $\Pi^{\text{an}}$. Accordingly, when moving to simulated quantities, we set

$$\widehat{\text{Std}}(\Pi_c(\gamma^*); n_p) = \widehat{\text{Std}}(\Pi; n_p)(1 - \widehat{\text{Corr}}(\Pi, \Pi^{\text{an}}; n_p)^2)^{1/2},$$

where $\widehat{\text{Corr}}(\Pi, \Pi^{\text{an}}; n_p)$ is the sample correlation

# Monte Carlo pricing : Control Variate

$$\widehat{\mathsf{Corr}}(\Pi, \Pi^{\mathsf{an}}; n_p) = \frac{\widehat{\mathsf{Cov}}(\Pi, \Pi^{\mathsf{an}}; n_p)}{\widehat{\mathsf{Std}}(\Pi; n_p)\ \widehat{\mathsf{Std}}(\Pi^{\mathsf{an}}; n_p)}$$

and the sample covariance is

$$\widehat{\mathsf{Cov}}(\Pi, \Pi^{\mathsf{an}}; n_p) = \sum_{j=1}^{n_p} \Pi^j \Pi^{\mathsf{an},j}/n_p - (\sum_{j=1}^{n_p} \Pi^j)(\sum_{j=1}^{n_p} \Pi^{\mathsf{an},j})/(n_p^2)$$

and

$$(\widehat{\mathsf{Std}}(\Pi^{\mathsf{an}}; n_p))^2 := \sum_{j=1}^{n_p} (\Pi^{\mathsf{an},j})^2/n_p - (\sum_{j=1}^{n_p} \Pi^{\mathsf{an},j}/n_p)^2.$$

# Monte Carlo pricing : Control Variate

One may include the correction factor $n_p/(n_p - 1)$ to correct for the bias of the variance estimator, although the correction is irrelevant for large $n_p$.

We see from

$$\widehat{\mathsf{Std}}(\Pi_c(\gamma^*); n_p) = \widehat{\mathsf{Std}}(\Pi; n_p)(1 - \widehat{\mathsf{Corr}}(\Pi, \Pi^{\mathsf{an}}; n_p)^2)^{1/2},$$

that for the variance reduction to be relevant, we need to choose the analytical payoff $\Pi^{\mathsf{an}}$ to be as (positively or negatively) correlated as possible with the original payoff $\Pi$. Notice that in the limit case of correlation equal to one the variance shrinks to zero.

The window for our control-variate Monte Carlo estimate $\widehat{\Pi}_c(\gamma; n_p)$ of $E(\Pi)$ is now:

$$\left[\widehat{\Pi}_c(\gamma; n_p) - 2.33\,\frac{\widehat{\mathsf{Std}}(\Pi_c(\gamma^*); n_p)}{\sqrt{n_p}},\quad \widehat{\Pi}_c(\gamma; n_p) + 2.33\,\frac{\widehat{\mathsf{Std}}(\Pi_c(\gamma^*); n_p)}{\sqrt{n_p}}\right],$$

This window is narrower than the corresponding simple Monte Carlo one by a factor $(1 - \widehat{\mathsf{Corr}}(\Pi, \Pi^{\mathsf{an}}; n_p)^2)^{1/2}$.

# Monte Carlo pricing : Control Variate

We may wonder about a good possible $\Pi^{\text{an}}$. We may select as $\Pi^{\text{an}}$ the simplest payoff depending on the underlying default indicators:

$$1_{\{\tau_1<T\}}, 1_{\{\tau_2<T\}}, \ldots, 1_{\{\tau_n<T\}}.$$

We may consider the product or the sum of the default or survival indicators. We will see later on in the multiname part that the expectation of these payoffs is known analytically in terms of chosen copula functions. In the single name case, we can simply take $\Pi^{\text{an}} = 1_{\{\tau<T\}}$, whose expectation is known analytically and is simply the probability of defaulting by time $T$ (as is for example stripped from CDS contracts).

# Stochastic Intensity. The SSRD model

We have seen in detail CDS calibration and MC simulation in presence of **deterministic** and **time varying** intensity or hazard rates, $\gamma(t)dt = \mathbb{Q}\{\tau \in dt | \tau > t, \mathcal{F}_t\}$

As explained, this accounts for credit spread structure but not for **volatility**.

The latter is obtained moving to stochastic intensity (Cox process). The deterministic function $t \mapsto \gamma(t)$ is replaced by a stochastic process $t \mapsto \lambda(t) = \lambda_t$ The Hazard function $\Gamma(t) = \int_0^t \gamma(u)du$ is replaced by the Hazard process (or cumulated intensity) $\Lambda(t) = \int_0^t \lambda(u)du$.

Recall that $\Lambda(\tau) = \xi$, std exponential RV independent of $\mathcal{F}$, and thus $\tau = \Lambda^{-1}(\xi)$.

Recall that the intensity and the hazard process are $\mathcal{F}_t$-measurable, i.e. they are known at a given time based on default-free market information at that time.

It is the default component $\xi$ that is independent of anything else and thus “impossible to predict”.

# The SSRD model: The interest-rate part

Interest Rates Notation: $P(t,T)$: Discount factor at time $t$ for maturity $T$;

$L(t,T)$ : Simply compounded spot (LIBOR) rate at time $t$ for maturity $T$;

$$r_t = \lim_{T \to t} L(t,T) : \text{ Short rate.}$$

One-factor short rate models assume the whole yield curve at time $T$, i.e. $T \mapsto L(t,T)$, to be characterized by its initial point $r_t$ through a function $P(t,T) = P(t,T;r_t)$.

Time–homogeneous short–rate models: $r_t = x_t$, $dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t$

Depending on $\mu$ and $\sigma$, $P(t,T) = \mathbb{E}_t\left[\exp\left(-\int_t^T r_s\, ds\right)\right] =: P^x(t,T;x_t,\alpha)$ can admit an analytic expression or not. If not, to go from $r$ to $P$ one needs numerical solution of PDEs (lattices, finite diff.,...)

# The SSRD model: The interest-rate part

Time homog. models: Dynamics of $r_t = x_t$ under the risk–neutral measure:

1. **Vasicek (1977):** $dx_t = k(\theta - x_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma)$.
2. **Cox-Ingersoll-Ross (CIR, 1985):**

$$dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad \alpha = (k, \theta, \sigma), \quad 2k\theta > \sigma^2 \ .$$

3. **Dothan / Rendleman and Bartter:**

$$dx_t = ax_t dt + \sigma x_t dW_t, \ (x_t = x_0 \ e^{(a - \frac{1}{2}\sigma^2)t + \sigma W_t}, \ \alpha = (a, \sigma)).$$

4. **Exponential Vasicek:**

$$x_t = \exp(z_t), \quad dz_t = k(\theta - z_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma).$$

# The SSRD model: The interest-rate part

Fitting **to Zero curve**: Find $\alpha = \alpha^*$ such that model curve $T \mapsto L^x(0, T, x_0; \alpha)$ is closest to mkt curve $T \mapsto L^M(0, T)$.

Typically the fit is poor both qualitatively and quantitat., and no parameters left to fit volatility structures (caps, swaptions).

**Then** models with time–dependent coeff. were introduced. Dynamics of $r_t = x_t$ under the risk–neutral measure:

1. **Ho-Lee:** $dx_t = \theta(t)\ dt + \sigma\ dW_t$.
2. **Hull-White (Extended Vasicek):** $dx_t = k(\theta(t) -\ x_t)dt + \sigma dW_t$.
3. **Hull-White (Extended CIR):** $dx_t = k(\theta(t) -\ x_t)dt + \sigma\ \sqrt{x_t}\ dW_t$.
4. **Black-Derman-Toy (Extended Dothan):** $x_t = x_0\ e^{u(t)+\sigma(t)W_t}$
5. **Black-Karasinski (Ext exp Vasicek):** $x_t = e^{z_t},\ \ dz_t = k\,[\theta(t) - z_t]\,dt + \sigma dW_t$.

Now parameters are used to fit volatility structures.

# The SSRD model: The interest-rate part

| **Reference Model** | **Dist** | **ABP** | **AOP** | **Multif** | **M-R** | $r > 0?$ |
|---|---|---|---|---|---|---|
| Vasicek | $\mathcal{N}$ | Yes | Yes | Yes | Yes | No |
| CIR | n.c. $\chi^2$ | Yes | Yes | Yes | Yes | Yes |
| Dothan | $e^{\mathcal{N}}$ | "Yes" | No | No | "Yes" | Yes |
| Exp. Vasicek | $e^{\mathcal{N}}$ | No | No | No | Yes | Yes |

**Classical extended models:**

Distribution (Distr) Analytical bond prices (ABP) Analytical bond–option prices (AOP)

Mean Reversion (MR) Tractable Multi Factor Extension (Multif)

| **Extended Model** | **Distr** | **ABP** | **AOP** | **Multif** | **M-R** | $r > 0?$ |
|---|---|---|---|---|---|---|
| Ho-Lee | $\mathcal{N}$ | Yes | Yes | Yes | No | No |
| Hull-White (Vas.) | $\mathcal{N}$ | Yes | Yes | Yes | Yes | No |
| Hull-White (CIR) | n.c. $\chi^2$ | No | No | No | Yes | Yes-but |
| BDT | $e^{\mathcal{N}}$ | No | No | No | Yes | Yes |
| Black Karasinski | $e^{\mathcal{N}}$ | No | No | No | Yes | Yes |
| CIR ??? | s.n.c. $\chi^2$ | Yes | Yes | Yes | Yes | Yes |

# The SSRD model: The interest-rate part

We have seen extensions of $dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t$ , obtained through time varying coefficients, $r_t = x_t, \quad dx_t = \mu(x_t;\alpha(t))dt + \sigma(x_t;\alpha(t))dW_t$ .
**Instead,** we propose the following alternative possibility (B & M (2001)):

$$r_t = x_t + \phi(t;\alpha), \quad dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t \ ,$$

with $x_0$ **a further parameter** we include augmenting $\alpha$. We have the following bond and option prices:

$$P^r(t,T,r_t;\alpha) = \mathbb{E}_t\left\{\exp\left[-\int_t^T(\phi(s;\alpha)+x_s)ds\right]\right\} = \exp\left[-\int_t^T \phi(s;\alpha)ds\right] P^x(t,T,x_t;\alpha),$$

$$C^r(0,T,s,N,K,r_0;\alpha) = \mathbb{E}_0\left\{\exp\left[-\int_0^T r_u du\right](NP^r(T,s,r_T;\alpha)-K)^+\right\}$$

$$= \exp\left[-\int_0^s \phi(u;\alpha)du\right] C^x(0,T,s,N,K\exp\left[\int_T^s \phi(u;\alpha)\ du\right], x_0^\alpha;\alpha)$$

How do we select $\alpha$ and $\phi(\cdot,\alpha)$?

# The SSRD model: selecting $\phi$: fitting the zero curve

$$r_t = x_t + \phi(t;\alpha), \quad dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t \ .$$

Fitting the initial term structure. Solve

$$P^r(0,T,r_t;\alpha) = P^M(0,T) \ \text{ for all } T, \quad \text{i.e.}$$

$$\exp\left[-\int_0^T \phi(s;\alpha)ds\right] P^x(0,T,x_0;\alpha) = P^M(0,T), \quad \text{and obtain}$$

$$\phi(t;\alpha) = -\frac{\partial}{\partial t}\ln\left(\frac{P^M(0,t)}{P^x(0,t,r_0;\alpha)}\right) =: -\mathbf{f^x(0,t,r_t};\alpha) + \mathbf{f^M(0,t)}.$$

$$\int_a^b \phi(u;\alpha)du = -[R^x(0,b) - R^M(0,b)]b + [R^x(0,a) - R^M(0,a)]a.$$

If we select this $\phi$, we fit the initial term structure, **no matter the value of** $\alpha$.

Choose $\alpha$ (including $x_0$) **to fit caps/floors or a few swaptions** prices given analytically in terms of zero–bond option prices $C^x(0,T,s,N,K\exp\left[\int_T^s \phi(u;\alpha)\ du\right], x_0^\alpha;\alpha)$.

# The SSRD Model. Extending CIR: The CIR++ model

$$dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad \alpha = (k, \theta, \sigma, x_0), \quad 2k\theta > \sigma^2 \ .$$

$x \sim$ noncentral chi-square; Important: $x > 0$. Here, due to non-linearity, our approach (CIR++: B.& M. (2001a,b))

$$r_t = x_t + \phi(t, \alpha),$$

is NOT equivalent to H-W-CIR $dr_t = k(\vartheta(t) - r_t)dt + \sigma\sqrt{r_t}\, dW_t$.

H-W: numerical problems, analytical tractability lost.

Instead, we keep **analytical tractability** and by restricting the parameters domain we can preserve **positive rates** while still **perfectly fitting** the yield curve.

**Advantages:** Positive rates. Analytical tractability. Low dimension. Known transition densities. Easy montecarlo and lattices. Chi-squared tails.

**Disadvantages**: Unrealistic volatility term-structure, possibly poor fitting to cap/floor prices (as with most short-rate models.)

# The SSRD Model. Some numerical results on CIR++

$r(t) = x(t) + \varphi^{CIR}(t;\alpha)$ has positive rates if

$$\varphi^{CIR}(t;\alpha) = f^M(0,t) - f^{CIR}(0,t;\alpha) > 0 \quad \text{for all } t \geq 0.$$

There are **sufficient conditions on the parameters** $\alpha = [k, \theta, \sigma, x_0]$ and guaranteeing this, but they can be **too restrictive** on caps/floors fitting. A good compromise is requiring

$$\boxed{\frac{2k\theta}{k+h} < \lim_{t \to +\infty} f^M(0,t), \quad x_0 < r_0} \quad (h = \sqrt{k^2 + 2\sigma^2})$$

which guarantees positivity only in most situations (in every practical case we tested).

# The SSRD Model: CIR++ stochastic intensity $\lambda$

We have taken CIR++ stochastic interest rates $r$, $r_t = x_t^\alpha + \phi(t, \alpha)$.

Now we model the intensity: consider

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t,\ \ \beta = (\kappa, \mu, \nu, y_0),\ \ 2\kappa\mu > \nu^2.$$

As before, $y \sim$ nonc. chi-square; Very important: $y > 0$. Set

$$\lambda_t = y_t^\beta + \psi(t;\beta)\ ,\quad t \geq 0,$$

where $\psi$ is a deterministic shift depending on the parameters $\beta$.

For restrictions on the $\beta$'s that keep $\lambda$ positive, **as is required in intensity models**, we may use the results in B. and M. (2001) summarized earlier for interest rates. We will often use the hazard process $\Lambda(t) = \int_0^t \lambda_s ds$, and also $Y(t) = \int_0^t y_s ds$ and $\Psi(t, \beta) = \int_0^t \psi(s, \beta) ds$.

# The SSRD Model: CIR++ stochastic intensity $\lambda$. Calibrating Implied Default Probabilities

If we can read from the market some implied risk-neutral default probabilities, and associate to them implied hazard functions $\Gamma^{\mathrm{Mkt}}$, we may wish our stochastic intensity model to agree with them. In a stochastic intensity model we have

$$\exp(-\Gamma^{\mathrm{Mkt}}(t)) = \mathbb{Q}\{\tau > t\} = \mathbb{Q}\{\Lambda(\tau) > \Lambda(t)\} = \mathbb{Q}\{\xi > \Lambda(t)\} =$$

$$\mathbb{E}1\{\xi > \Lambda(t)\} = \mathbb{E}[\mathbb{E}\{1\{\xi > \Lambda(t)\}|\mathcal{F}^{\lambda}\}] = \mathbb{E}[e^{-\Lambda(t)}] = \mathbb{E}[e^{-\int_0^t \lambda(s)ds}]$$

$$= \mathbb{E}[\exp\{-\int_0^t (y_s + \psi(s,\beta))ds\}] = \mathbb{E}\exp\left(-\Psi(t,\beta) - Y(t)\right)$$

$$= \exp\left(-\Psi(t,\beta)\right)\mathbb{E}\exp\left(-Y(t)\right) = \exp\left(-\Psi(t,\beta)\right)\mathbb{E}[e^{-\int_0^t y_s ds}]$$

IMPORTANT 1: The second equality is possible only if $\lambda$ is strictly positive;

IMPORTANT 2: It is fundamental, if we aim at calibrating default probabilities, that the last expected value can be computed analytically.

**The only diffusion model satisfying both constraints is CIR++**

# The SSRD Model: CIR++ stochastic intensity $\lambda$. Calibrating Implied Default Probabilities

$$\exp(-\Gamma^{\mathsf{Mkt}}(t)) = \mathbb{Q}\{\tau > t\} = \exp\left(-\Psi(t,\beta)\right)\mathbb{E}[e^{-\int_0^t y_s ds}]$$

Now notice that $\mathbb{E}[e^{-\int_0^t y_s ds}]$ is simply the bond price for a CIR interest rate model with short rate given by $y$, so that it is known analytically. We denote it by $P^y(0,t,y_0;\beta)$.

Similarly to the interest-rate case, $\lambda$ is calibrated to the market implied hazard function $\Gamma^{\mathsf{Mkt}}$ if we set

$$\Psi(t,\beta) := \Gamma^{\mathsf{Mkt}}(t) + \ln(P^y(0,t,y_0;\beta))$$

where we choose the parameters $\beta$ in order to have a positive function $\psi$, by resorting to the condition seen earlier.

Piecewise linear hazard rate $\gamma(t)$ (deterministic intensity) obtained by calibrating the 1y, 3y, 5y, 7y and 10y CDS's on Merrill-Lynch on October 2002.

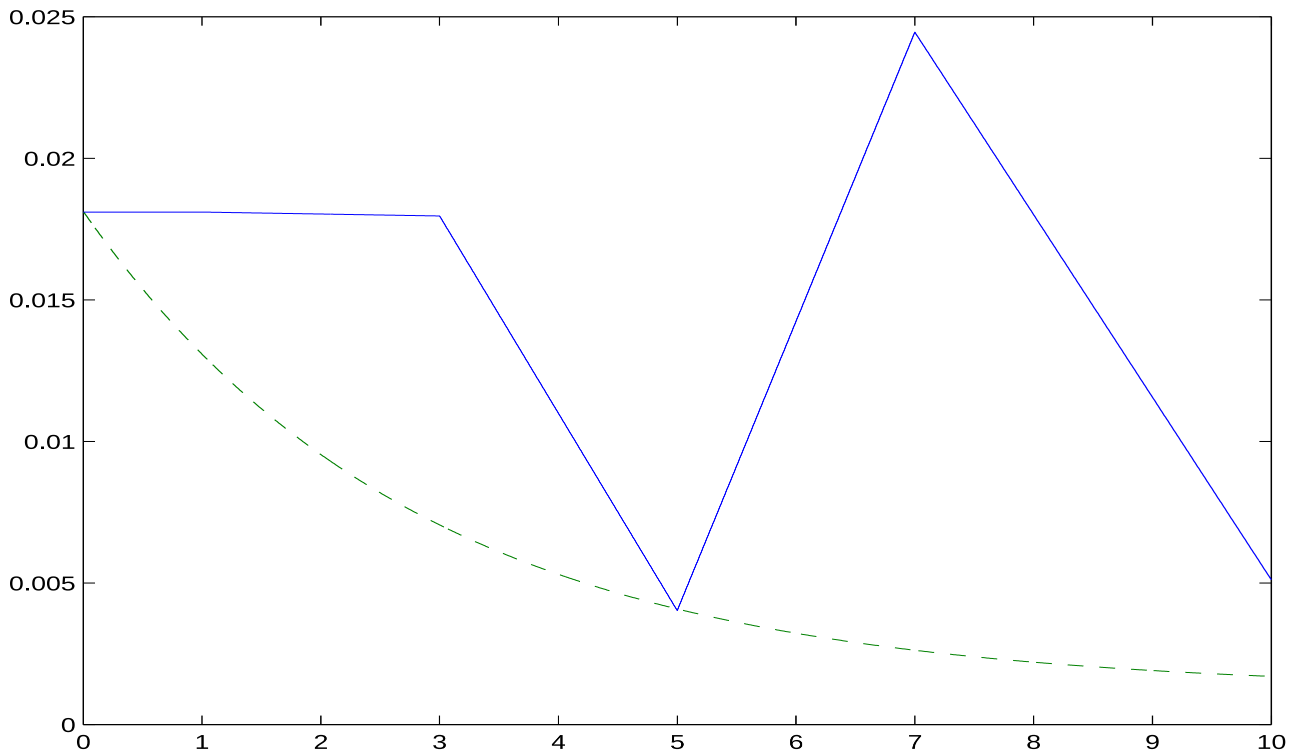


Figure 21: Intensity for Merrill-Lynch CDS's of several maturities on October 25, 2002

Piecewise quadratic hazard function $\Gamma(t)$ and related risk neutral default probability ($\mathbb{Q}\{\tau < t\} = 1 - \exp(-\Gamma(t)) \approx \Gamma(t)$ for small $\Gamma$) obtained by calibrating the 1y, 3y, 5y, 7y and 10y CDS's on Merrill-Lynch on October 2002

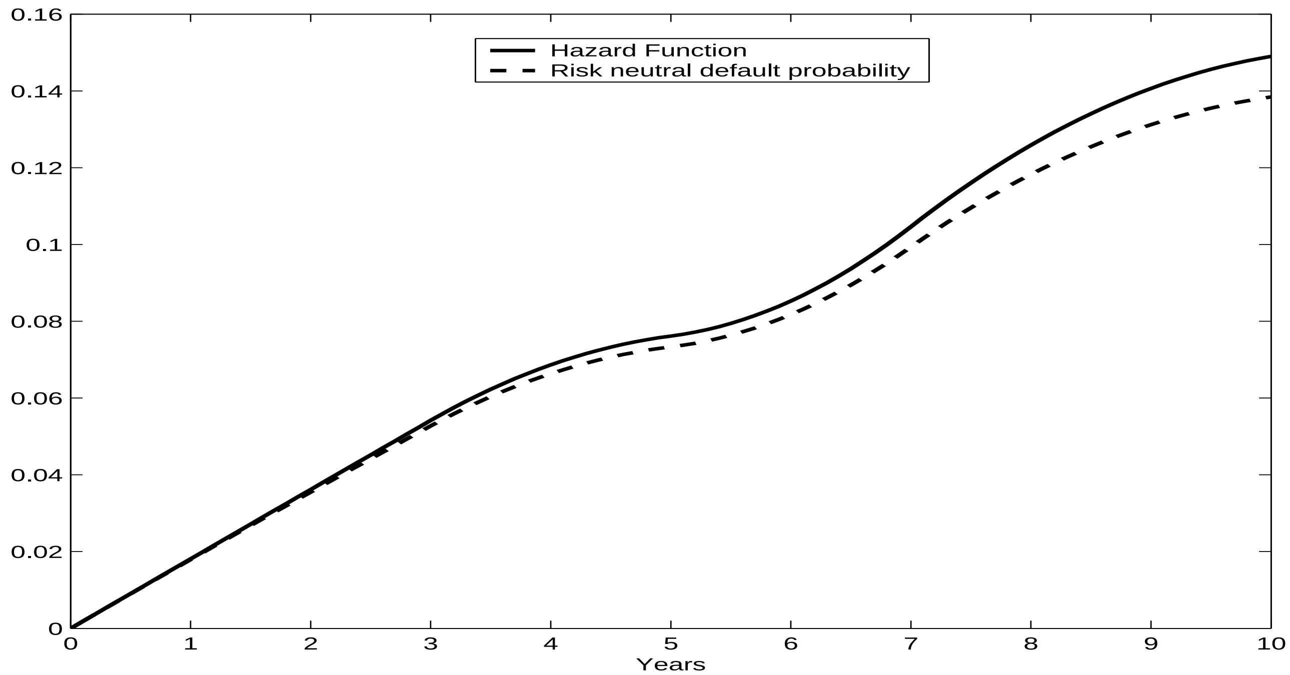


Figure 22: Hazard function and risk-neutral default probability for ML CDS's on October 25, 2002

Calibrating to the ML data up to 5y gives us the following parameters and $\psi$. To select the values of $\beta$ in the CIR dynamics for $y$ the minimization imposes positivity of $\psi$ and gives the smallest integral: $\beta^* :=$ argmin$_\beta \int_0^5 \psi(u;\beta)^2 du$.
Later on $\beta$, **which is free**, can be used to calibrate CDS **option** data.

$$\beta : \quad \kappa = 0.354201,\ \mu = 0.00121853,\ \nu = 0.0238186;\ y_0 = 0.0181,$$

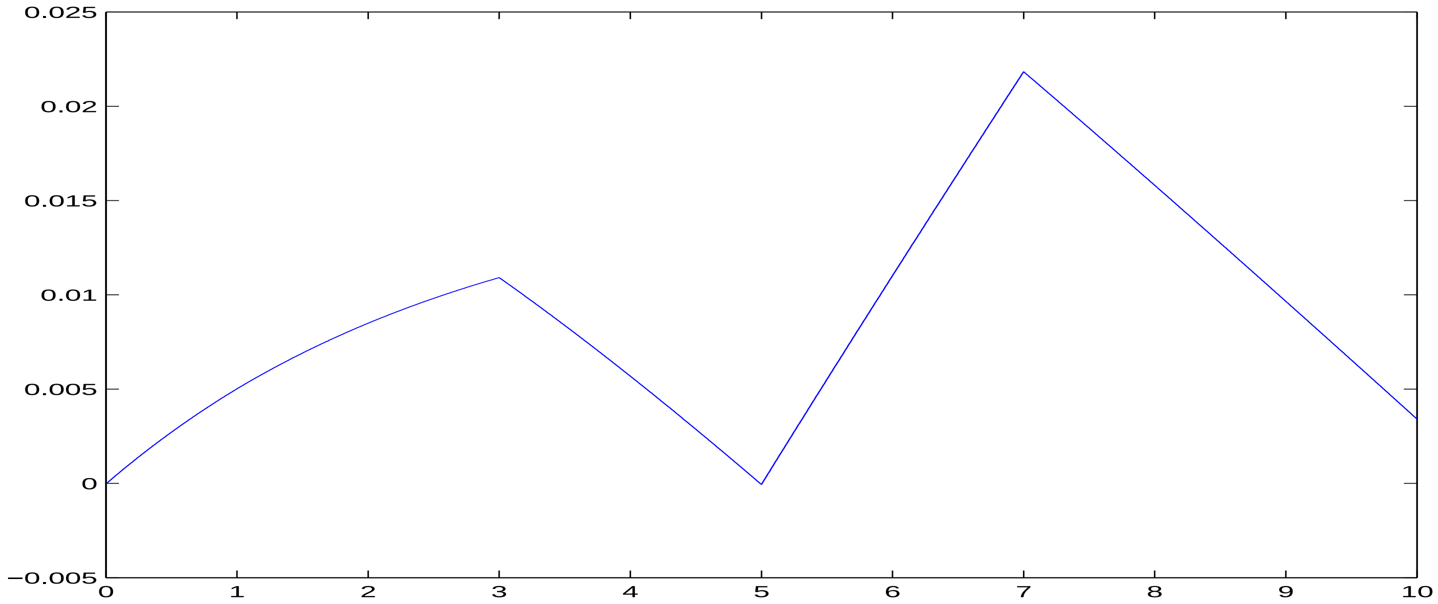


Figure 23: $\psi$ function for the CIR++ model for $\lambda$ calibrated to Merrill-Lynch CDS's up to 5y

# The SSRD Model: Stochastic Intensity and Stochastic Rates

Now we can consider the fully stochastic model for interest rates and intensities:

$$
\begin{aligned}
dx_t &= k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad \alpha = (k, \theta, \sigma, x_0), \quad 2k\theta > \sigma^2, \\
dy_t &= \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t, \ \beta = (\kappa, \mu, \nu, y_0), \ 2\kappa\mu > \nu^2, \ \ dZ\ dW = \rho\ dt \\
r_t &= x_t^\alpha + \phi(t;\alpha), \quad \lambda_t = y_t^\beta + \psi(t;\beta).
\end{aligned}
$$

interest rate model: analytical formulas (Calibration) for the zero curve and caps;
intensity model: analytical formulas (Calibration) for risk-neutral default probabilities;

The two models are correlated.

Calibrate $r$ to interest-rate curve and caps, $\lambda$ to CDS's implied hazard functions $\Gamma^{\mathsf{Mkt}}$;

This separate calibration implicitly assumes that $\rho = 0$. It can be shown that this is consistent to directly calibrate the stochastic model to CDS prices when $\rho = 0$.

To check the impact of correlation on CDS we may calibrate the model with $\rho = 0$, then set a $\rho \neq 0$ and reprice the same CDS, checking the corresponding price change.

## Stochastic Intensity and Stochastic Rates: Impact of $\rho$ on CDS's

$$\text{CDS}_{0,b}(0,R,\text{LGD}) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\Big\{ D(0,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{\tau<T_b\}} + \sum_{i=1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{\tau<T_b\}}D(t,\tau)\ \text{LGD} \Big\}$$

In our intensity SSRD model $\tau = \Lambda^{-1}(\xi)$, with $\xi$ unit exponential random variable independent of $\lambda$ and $r$, with $\lambda$ and $r$ correlated CIR++ processes, $\Lambda(t) = \int_0^t \lambda(s)ds$.

The market quoted $R$ renders the initial 5y CDS fair,

$\text{CDS}_{0,5y}(0, R^{\text{MID}}_{0,5y}(0), \text{LGD}) = 0$ with the deterministic model or with the stochastic model with $\rho = 0$.

$R^{\text{BID}}_{0,5y}(0) = 0.009$, $R^{\text{ASK}}_{0,5y}(0) = 0.0098$, $R^{\text{MID}}_{0,5y}(0) = 0.0094$, $\text{LGD} = 0.593$;

# Stochastic $\lambda$ and $r$: impact of $\rho$

By Monte Carlo simulation (Gaussian mapping) of the joint CIR processes and with a large number of scenarios we obtain, with the above calibration (notional is $1$)

| CDS prices | Gaussian Mapping | Monte Carlo value and $95\%$ window |
|---|---|---|
| $\rho = -1$ | -1.12E-4 | -1.48625E-4 (-1.79586 -1.17664) |
| $\rho = 0$ | 0.012E-4 | 0.17708E-4 (-0.142444 0.496605) |
| $\rho = 1$ | 1.14E-4 | 1.25475E-4 (0.922997 1.5865) |

Same run with $\kappa, \nu$ increased by a factor $5$ and $\mu$ by a factor $3$ :

| CDS prices | Gaussian Mapping | Monte Carlo value and $95\%$ window |
|---|---|---|
| $\rho = -1$ | -1.03E-4 | -1.77E-4 (-2.02 -1.51) |
| $\rho = 0$ | 0.021E-4 | 0.143E-4 (-0.138 0.424) |
| $\rho = 1$ | 1.07E-4 | 1.08E-4 (0.78 1.37) |

Table 17: 5y CDS prices as a function of $\rho$ with MC simulation

The deterministic model prices ($\rho = 0$) with $R^{\text{BID}}$ and $R^{\text{ASK}}$ are CDS$^{\text{BID}}$ =-17.14 E-4, CDS$^{\text{ASK}}$ =17.16 E-4, so **correlation yields an effect that is about 1/10 the bid-ask spread** in the deterministic model.

# Stochastic $\lambda$ and $r$: impact of $\rho$

It seems than that we may freely calibrate the model to CDS by assuming $\rho = 0$, i.e. by separately calibrating rates and intensities to $(\alpha, \phi(\cdot;\alpha))$ and $(\beta, \psi(\cdot,\beta))$ respectively. We then set $\rho$ to a desired value $\neq 0$.

The inconsistency of this procedure as concerns CDS prices reproduced by the model is typically negligible. However, $\rho$ may have a measurable impact on products with stronger nonlinearity than CDS's (cancellable swaps, tranches of some defaultable structures)
Examples: Consider the following terms with the hazard function data as before and $r$ calibrated to typical zero-curve and cap interest-rate data.

$$A = D(0,5y)L(4y,5y)\mathbf{1}_{\tau<5y}, \quad B = D(0,\tau)\mathbf{1}_{\tau<5y}$$

$$C = D(0,\min(\tau,5y)), \quad D = D(0,5y)L(4y,5y)\mathbf{1}_{\tau\in[4y,5y]},$$

# Stochastic $\lambda$ and $r$: impact of $\rho$

$$A = D(0,5y)L(4y,5y)\mathbf{1}_{\tau<5y}, \quad B = D(0,\tau)\mathbf{1}_{\tau<5y}$$

$$C = D(0,\min(\tau,5y)), \quad D = D(0,5y)L(4y,5y)\mathbf{1}_{\tau\in[4y,5y]},$$

These payoffs appear typically in basic credit derivatives. Traders may check the impact of correlation on different products.

| | $\rho = -1$ | $\rho = 1$ | rel variation | abs variation |
|---|---|---|---|---|
| A | 30.3672 bps | 31.1962 | +2.73% | +0.829 |
| B | 679.197 bps | 676.208 | -0.44% | -2.989 |
| C | 8207.23 bps | 8209.61 | +0.03% | +2.38 |
| D | 2.77376 bps | 3.10889 | +10.77% | +0.34 |

# Stochastic $\lambda$ and $r$: CDS Options

Recall a CDS option is the option to enter at a future time $T_a$ into a CDS protecting up to time $T_b > T_a$ at a given premium rate in the leg $K$.

$$\mathrm{CDS}_{a,b}(T_a, K, \mathrm{L_{GD}}) = 1_{\{\tau > T_a\}} \mathbb{E}\Big\{ D(T_a,\tau)(\tau - T_{\beta(\tau)-1}) K \mathbf{1}_{\{\tau < T_b\}}$$

$$+ \sum_{i=a+1}^{b} D(T_a, T_i)\alpha_i K \mathbf{1}_{\{\tau > T_i\}} - \mathbf{1}_{\{\tau < T_b\}} D(T_a,\tau)\, \mathrm{L_{GD}} | \mathcal{G}_{T_a} \Big\}$$

$$= 1_{\{\tau > T_a\}} \Big\{ K \int_{T_a}^{T_b} \mathbb{E}\left[ \exp\left( -\int_{T_a}^{u} (r_s + \lambda_s) ds \right) \lambda_u | \mathcal{F}_{T_a} \right] (u - T_{\beta(u)-1}) du$$

$$+ K \sum_{i=a+1}^{b} \alpha_i \mathbb{E}\left[ \exp\left( -\int_{T_a}^{T_i} (r_s + \lambda_s) ds \right) | \mathcal{F}_{T_a} \right]$$

$$- \mathrm{L_{GD}} \int_{T_a}^{T_b} \mathbb{E}\left[ \exp\left( -\int_{T_a}^{u} (r_s + \lambda_s) ds \right) \lambda_u | \mathcal{F}_{T_a} \right] du \Big\}$$

$$:= 1_{\{\tau > T_a\}} \mathrm{CDS}_{\mathcal{F}}(T_a, T_a, T_b, K, \mathrm{L_{GD}}; x_{T_a}, y_{T_a}).$$

# Stochastic $\lambda$ and $r$: CDS Options

**Assuming $\rho = 0$ from $T_a$ on**, the expectations appearing in the above expression can be computed as follows:

$$\mathbb{E}\left[\exp\left(-\int_{T_a}^{T_i}(r_s+\lambda_s)ds\right)|\mathcal{F}_{T_a}\right] = \exp(\Psi(T_a,\beta)-\Psi(T_i,\beta))P^{\mathsf{CIR}}(T_a,T_i;y_{T_a},\beta)\times$$
$$\times\exp(\Phi(T_a,\alpha)-\Phi(T_i,\alpha))P^{\mathsf{CIR}}(T_a,T_i;x_{T_a},\alpha).$$

Further, we may compute

$$\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}(r_s+\lambda_s)ds\right)\lambda_u|\mathcal{F}_{T_a}\right] = \mathbb{E}\left[e^{-\int_{T_a}^{u}r_s ds}|\mathcal{F}_{T_a}\right]\mathbb{E}\left[e^{-\int_{T_a}^{u}\lambda_s ds}\lambda_u|\mathcal{F}_{T_a}\right] =$$
$$= \mathbb{E}\left[\exp\left(-\int_{T_a}^{u}r_s ds\right)|\mathcal{F}_{T_a}\right]\left(-\frac{d}{du}\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}\lambda_s ds\right)|\mathcal{F}_{T_a}\right]\right) =$$
$$= -e^{\Phi(T_a,\alpha)-\Phi(u,\alpha)}P^{\mathsf{CIR}}(T_a,u;x_{T_a},\alpha)\frac{d}{du}\left[e^{\Psi(T_a,\beta)-\Psi(u,\beta)}P^{\mathsf{CIR}}(T_a,u;y_{T_a},\beta)\right]$$

# Stochastic $\lambda$ and $r$: CDS Options

so that all terms are known analytically given the simulated paths of $x_{T_a}$ and $y_{T_a}$, which are to be simulated with nonzero $\rho$ from time $0$ to time $T_a$. Putting all pieces together, without forgetting the indicator $1_{\{\tau>T_a\}}$, we may value the CDS option payoff by simulation.

$$\mathbb{E}\left[D(t,T_a)[-\mathrm{CDS}_{a,b}(T_a,K,\mathrm{L}_{\mathrm{GD}})]^+|\mathcal{G}_t\right]$$

$$= \mathbb{E}\left[D(t,T_a)1_{\{\tau>T_a\}}[-\mathrm{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\mathrm{L}_{\mathrm{GD}};x_{T_a},y_{T_a})]^+|\mathcal{G}_t\right]$$

$$= \frac{1_{\{\tau>t\}}}{\exp(-\Lambda(t))}\mathbb{E}\left[D(t,T_a)1_{\{\tau>T_a\}}[-\mathrm{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\mathrm{L}_{\mathrm{GD}};x_{T_a},y_{T_a})]^+|\mathcal{F}_t\right]$$

$$= 1_{\{\tau>t\}}\mathbb{E}\left[D(t,T_a)\exp(-\Lambda(T_a)+\Lambda(t))[-\mathrm{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\mathrm{L}_{\mathrm{GD}};x_{T_a},y_{T_a})]^+|\mathcal{F}_t\right]$$

$$= \boxed{1_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^{T_a}(r_s+\lambda_s)ds\right)[-\mathrm{CDS}_{\mathcal{F}}(T_a,T_a,T_b,K,\mathrm{L}_{\mathrm{GD}};x_{T_a},y_{T_a})]^+|\mathcal{F}_t\right]} \quad (7)$$

## Stochastic $\lambda$ and $r$: CDS Options and impact of $\rho$

Valuing this contract with the CIR++ model when $\rho \neq 0$ from $T_a$ on can be a problem, since we have no closed form formula for $\bar{P}$ or the other terms at time $T_a$.

We would thus be forced, in principle, to sub-simulate paths from $T_a$ to $T_b$ just to be able to obtain the underlying asset $\text{CDS}_{a,b}(T_a)$ of the option at $T_a$.

This is computationally undesirable and we need to find alternatives.

Here are some tests on the impact of $\rho$ and on the possibility to set $\rho = 0$ from $T_a$ on.

# Stochastic $\lambda$ and $r$: CDS Options and impact of $\rho$

| Time | $\gamma^{\text{mkt}}$ | Survival Probability |
|---|---|---|
| 26-Jun-03 | 0.0374016 | 1 |
| 28-Jun-04 | 0.0413386 | 0.96108375 |
| 27-Jun-06 | 0.0442196 | 0.882378007 |
| 27-Jun-08 | 0.0446496 | 0.807246935 |

Table 18: Parmalat

We take into account the Euro default free interest rate curve of the same day and set in all simulations, the $\alpha$ parameters of the EURO interest-rate curve to

$$k = 0.4, \theta = 0.026, \sigma = 14\%, x_0 = 0.0165,$$

reflecting a possible calibration to Cap volatilities.
As concerns the Monte Carlo method, all the following simulations are obtained by means of 50,000 paths, under variance reduction techniques, for the relevant stochastic processes $x$ and $y$. When the correlation is not zero after $T_a$, the CDS prices at $T_a$ are approximated with 5,000 paths .

# Stochastic $\lambda$ and $r$: CDS Options and impact of $\rho$

| | |
|---|---|
| $T_a$ | 1 year |
| $T_b$ | 5 years |
| $T_{a+i+1} - T_{a+i}$ | 6 months |
| $K$ | 311 bp |
| LGD | 70 % |

| | |
|---|---|
| $T_a$ | 4 years |
| $T_b$ | 5 years |
| $T_{a+i+1} - T_{a+i}$ | 6 months |
| $K$ | 319 bp |
| LGD | 70 % |

**Impact of the correlation from $T_a$ on**: The results obtained for the 1 year/4 years Parmalat CDS option and different values of $\rho_{0,T_a}$ and $\rho_{T_a,T_b}$ ($\kappa = 0.5$, $\mu = 0.0475$, $\nu = 20\%$, $y_0 = 0.037$ being fixed) are summarized in the next table.

| impl volatility | $\rho_{T_a,T_b} = -1$ | $\rho_{T_a,T_b} = 0$ | $\rho_{T_a,T_b} = 1$ |
|---|---|---|---|
| $\rho_{0,T_a} = -1$ | 30.7 (29.2 ; 32.2) % | 31.0 (30.9 ; 31.2) % | — |
| $\rho_{0,T_a} = 0$ | 31.7 (30.2 ; 33.2) % | 30.4 (30.2 ; 30.6) % | 30.0 (28.5 ; 31.4) % |
| $\rho_{0,T_a} = 1$ | — | 29.4 (29.2 ; 29.5) % | 24.8 (23.3 ; 26.2) % |

## Stochastic $\lambda$ and $r$: CDS Options and impact of $\rho$

**Impact of the correlation from $T_a$ on**. The results obtained for the 4 years/1 year Parmalat CDS option and different values of $\rho_{0,T_a}$ and $\rho_{T_a,T_b}$ ($\kappa = 0.5$, $\mu = 0.0475$, $\nu = 20\%$, $y_0 = 0.037$ being fixed) are summarized in the next table.

| impl volatility | $\rho_{T_a,T_b} = -1$ | $\rho_{T_a,T_b} = 0$ | $\rho_{T_a,T_b} = 1$ |
|---|---|---|---|
| $\rho_{0,T_a} = -1$ | 33.4 (32.7 ; 34.2) % | 34.7 (34.5 ; 34.9) % | — |
| $\rho_{0,T_a} = 0$ | 33.0 (32.3 ; 33.7) % | 33.4 (33.2 ; 33.7) % | 33.3 (32.6 ; 34.1) % |
| $\rho_{0,T_a} = 1$ | — | 31.4 (31.2 ; 31.7) % | 30.7 (30.0 ; 31.4) % |

So we may ignore correlation between interest rates and stochastic intensity, thus simplifying monte carlo, only when said correlation is negative (as is typically deemed to be in traditional works).

# Stochastic $\lambda$ and $r$: Gaussian mapping

Since correlated CIR(++) processes are not tractable, we use a Gaussian mapping approximation. We consider a Vasicek model with the same drift coefficients in the processes $x$ and $y$ and look for diffusion coefficients (vols) such that (notation is obvious from the context)

$$\mathbb{E}\left[\exp\left(-\int_0^T x_s^{\alpha T,V} ds\right)\right] = \mathbb{E}\left[\exp\left(-\int_0^T x_s^{\alpha} ds\right)\right] \; (\mathbb{E}\left[\exp\left(-\int_0^T y_s^{\beta T,V} ds\right)\right] = \mathbb{E}\left[\exp\left(-\int_0^T y_s^{\beta} ds\right)\right] \; )$$

and then use the following known expressions to value payoffs:

$$\mathbb{E}\left[\exp\left(-\int_0^T (x_s^{\alpha T,V} + y_s^{\beta T,V}) ds\right)\right], \quad \mathbb{E}\left[\exp\left(-\int_0^T (x_s^{\alpha T,V} + y_s^{\beta T,V}) ds\right) y_T^{\beta T,V}\right] + \Delta$$

where $\Delta =$

$$\mathbb{E}\left[\exp\left(-\int_0^T x_s^{\alpha} ds\right)\right] \mathbb{E}\left[\exp\left(\int_0^T y_s^{\beta} ds\right) y_T^{\beta}\right] - \mathbb{E}\left[\exp\left(-\int_0^T x_s^{\alpha,T,V} ds\right)\right] \mathbb{E}\left[\exp\left(\int_0^T y_s^{\beta,T,V} ds\right) y_T^{\beta,T,V}\right]$$

and where we use the known analytical expressions for the right-hand sides, where the two Vasicek processes have the same $\rho$ as the original CIR processes. Tested with MC simulation. OK. Details in Brigo and Alfonsi (2003).

# Implicit positivity-preserving scheme for MC

In presence of correlation, joint distribution of correlated CIR processes $x$ and $y$ is unknown, and in particular distribution of $x_t + y_t$ is unknown.

In general we need MC simulation to value payoffs. Can use Euler or Milstein schemes for $(x, y)$, but these do not ensure well behaved processes (some steps can lead to negative and then imaginary $x$'s or $y$'s).

A remedy is refining the discretization between such steps via a Brownian bridge, thus re-gaining positivity by approaching the continuous process, which we know to remain positive.

A more elegant remedy consists in designing an implicit Euler scheme that can be proven to maintain positivity of $x$ and $y$ and that has the same order of convergence of the explicit scheme.

Details in Brigo and Alfonsi (2003).

# Numerical tests

$$\mathbb{E}\left[\exp\left(-\int_0^T (x_s^{\alpha_T,V} + y_s^{\beta_T,V})ds\right)\right], \quad \mathbb{E}\left[\exp\left(-\int_0^T (x_s^{\alpha_T,V} + y_s^{\beta_T,V})ds\right) y_T^{\beta_T,V}\right] + \Delta$$

We compare these expected values with $T = 5y$ when $(x, y)$ are the MC simulated true CIR processes

$$\alpha : \quad k = 0.528905, \ \theta = 0.0319904, \ \ \sigma = 0.130035, \ x_0 = 8.32349 \times 10^{-5}.$$

$$\beta : \quad \kappa = 0.354201, \ \mu = 0.00121853, \ \nu = 0.0238186; \ y_0 = 0.0181),$$

with the expected values obtained via mapped Vasicek processes (the Vasicek mapped volatilities are $\sigma^{V,5y} = 0.016580$ and $\nu^{V,5y} = 0.0025675$).

| first approx | $\rho =$ -1 | $\rho =$1 |
|---|---|---|
| MC for (8) | 0.86191 (0.861815 0.862004) | 0.8624 ( 0.862272 0.862529) |
| Appr (8) | 0.861762, | 0.862554 |

If the values in the Table were interpreted as bonds, the continuously compounded spot rates would be $-\ln(0.86191)/5 = 0.02972$ and $-\ln(0.861762)/5 = 0.029755$, giving a small difference.

| second approx | $\rho =$ -1 | $\rho =$1 |
|---|---|---|
| MC for (8) | 3.5848E-3 (3.57946 3.59014) | 3.44852E-3 (3.44408 3.45295) |
| Appr (8) | 3.59831E-3 | 3.43174E-3 |

# Numerical tests

$$\mathbb{E}\left[\exp\left(-\int_0^T (x_s^{\alpha_T,V} + y_s^{\beta_T,V})ds\right)\right], \quad \mathbb{E}\left[\exp\left(-\int_0^T (x_s^{\alpha_T,V} + y_s^{\beta_T,V})ds\right) y_T^{\beta_T,V}\right] + \Delta$$

To check the approximation under stress, we multiply all parameters $k, \theta, \sigma$ and $\kappa, \mu, \nu$ by three and check again the approximation. Now the Vasicek mapped volatilities are $\sigma^{V,5y} = 0.108596$ and $\nu^{V,5y} = 0.0060675$.

| first approx | $\rho =$ -1 | $\rho =$1 |
|---|---|---|
| MC for (8) | 0.64232 (0.642106 0.642534) | 0.644151 (0.643909 0.644393) |
| Appr (8) | 0.641989 | 0.643904 |

If the values in the Table were interpreted as bond prices, the corresponding spot rates would be $-\ln(0.64232)/5 = 0.088534$ and $-\ln(0.641989)/5 = 0.088637$, with a larger difference than before, ranging around 1 basis point, which is however still contained.

| second approx | $\rho =$ -1 | $\rho =$1 |
|---|---|---|
| MC for (8) | 2.4757E-3 (2.46991 2.48149) | 2.27465E-3 (2.27018 2.27913) |
| Appr (8) | 2.53527 | 2.24435 |

So we may trust the approximation to work well within the typical market bid-ask spreads for CDS's. We have seen earlier how the approximation works for CDS's.

# Market Models for CDS Options

Recall the definition of CDS forward rate $R_{a,b}(t)$ for protection in $[T_a, T_b]$ as that rate in the premium leg of a (forward start) CDS protecting in $[T_a, T_b]$ that makes the forward CDS value equal to zero at the valuation time $t$. Since
CDS = R AccruedTerm (Accrued Premium) + R Annuity(premium leg) - ProtectionLeg,
we have R = ProtectionLeg/(AccruedTerm + Annuity)

$$0 = \boxed{R}\,\mathbb{E}[D(t,\tau)(\tau - T_{\beta(\tau)-1})\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t] +$$

$$+\boxed{R}\sum_{i=a+1}^{b} \alpha_i \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{F}_t] - \text{LGD}\,\mathbb{E}[\mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\ |\mathcal{F}_t]$$

where we have used the filtration switching formula to value the CDS given $\mathcal{F}_t$ rather than $\mathcal{G}_t$ and we have set to zero only the part without $1_{\{\tau>t\}}$.

This way our $R$ will be defined in every path and not only conditional on $\tau > t$.

# Market Models for CDS Options

Solving in $R$ we get, with the postponed payoff avoiding the accruing term,

$$0 = \boxed{R} \sum_{i=a+1}^{b} \alpha_i \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau > T_i\}} | \mathcal{F}_t] - \mathsf{L}\text{GD} \sum_{i=a+1}^{b} \mathbb{E}[\mathbf{1}_{\{T_{i-1} < \tau \le T_i\}} D(t,T_i) \, | \mathcal{F}_t]$$

leading to

$$R_{a,b}^{PR}(t) = \frac{\mathsf{L}\text{GD} \ \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1} < \tau \le T_i\}} | \mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau > T_i\}} | \mathcal{F}_t]}$$

or, by using the filtration switching formula backwards in the denominator and recalling the definition of defaultable zero coupon bond $\bar{P}$,

$$R_{a,b}^{PR}(t) = \frac{\mathsf{L}\text{GD} \ \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1} < \tau \le T_i\}} | \mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t | \mathcal{F}_t) \bar{P}(t,T_i)}$$

# Market Models for CDS Options

$$R^{PR}_{a,b}(t) = \frac{\text{L}_{\text{GD}} \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)}$$

Consider again the **option** to enter a CDS at the future time $T_a$ paying a fixed rate $K$ at times $T_{a+1}, \ldots, T_b$ or until default, in exchange for a protection payment LGD against possible default in $[T_a, T_b]$ (payer CDS option).

We have seen before that the payoff of this option can be written in terms of the fair CDS premium rate $R_{a,b}(T_a)$ prevailing at time $T_a$ as

$$\Pi\text{CallCDS}_{a,b}(t;K) = 1_{\{\tau>T_a\}} D(t,T_a) \left[\sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a,T_i)\right] (R_{a,b}(T_a) - K)^+ :$$

i.e. a Call option on $R_{a,b}(T_a)$.

# Market Models for CDS Options

$$\Pi\text{CallCDS}_{a,b}(t;K) = 1_{\{\tau>T_a\}} D(t,T_a) \left[\sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a,T_i)\right] (R_{a,b}(T_a) - K)^+$$

These options have been introduced here for postponed CDS, but can be interpreted also as options on the exact CDS payoff under an approximation (neglecting the $(\tau - T_{\beta(\tau)-1})$ term).

The quantity [.] is called ("no survival-indicator"-) "Defaultable Annuity". Actually the real Defaultable Annuity would have a $1_{\{\tau>\ \cdot\}}$ term in front of the summation. More generally, at time $t$, we set

$$\widehat{C}_{a,b}(t) := \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{C}_{a,b}(t), \quad \bar{C}_{a,b}(t) := \sum_{i=a+1}^{b} \alpha_i \bar{P}(t,T_i).$$

When including as a factor the indicator $1_{\{\tau>t\}}$, this quantity is the price, at time $t$, of a portfolio of defaultable zero-coupon bonds with zero recovery and with different maturities, and as such it is the price of a tradable asset, hence a possible **numeraire**.

# CDS Options and Callable Defaultable Floaters

As usual, one may wish to introduce implied volatility for CDS options. This would be a volatility associated to the relevant underlying CDS rate $R$.

In order to do so rigorously, one has to come up with an appropriate dynamics for our $R_{a,b}$ above directly, rather than modeling instantaneous default intensities $\lambda$ explicitly. This is done in a Cox process setting where we do not model directly stochastic intensity but rather some secondary market quantities that depend on it. This parallels the default-free interest rate market when we resort to the swap market model as opposed for example to a one-factor short-rate model for pricing swaptions.

| | | | |
|---|---|---|---|
| Fixed Income | inst. interest rate $r$ | vs | swap rate $S_{a,b}$ |
| $\updownarrow$ | $\updownarrow$ | | $\updownarrow$ |
| Credit | inst. intensity $\lambda$ | vs | CDS rate $R_{a,b}$ |

# CDS Options and Callable Defaultable Floaters

In the case of CDS options the market model is derived as follows. Consider as example the PR1CDS formulation. Take as numeraire the defaultable annuity $\widehat{C}_{a,b}$, so that

$$R^{PR}_{a,b}(t) = \frac{\text{Lgd} \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)} = \frac{\text{Lgd} \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}|\mathcal{F}_t]}{\widehat{C}_{a,b}(t)},$$

$t \leq T_a$ is a martingale under the measure associated to the annuity, since it is a tradable asset divided by the numeraire. This martingale can be modeled as a Black-Scholes (BS) driftless (martingale) geometric Brownian motion, leading to a BS formula for CDS options.

## CDS Options: Market Models (embedded Stochastic Intensity)

We denote here $R^{PR}$ by $R$. Compute, by resorting to the change of numeraire, iterated conditioning and switching from $\mathcal{G}$ to $\mathcal{F}$ (Payer CDS option):

$$\text{CallCDS}_{a,b}(t, K; \text{LGD}) = \mathbb{E}\{1_{\{\tau > T_a\}} D(t, T_a) \textstyle\sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a, T_i)(R_{a,b}(T_a) - K)^+ | \mathcal{G}_t\} = \dots$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}\big[D(t, T_a) \boxed{\widehat{C}_{a,b}(T_a)(R_{a,b}(T_a) - K)^+} |\mathcal{F}_t\big] = \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}^B\big[\frac{B(t)}{B(T_a)} \boxed{\text{Payoff}}\big] =$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}^{\widehat{C}}\big[\frac{\widehat{C}(t)}{\widehat{C}(T_a)} \boxed{\text{Payoff}}\big] = \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \widehat{C}(t) \mathbb{E}^{\widehat{C}}\big[\frac{\text{Payoff}}{\widehat{C}(T_a)}\big] =$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \widehat{C}_{a,b}(t) \widehat{\mathbb{E}}^{a,b}[(R_{a,b}(T_a) - K)^+ | \mathcal{F}_t] = 1_{\{\tau>t\}} \bar{C}_{a,b}(t) \widehat{\mathbb{E}}^{a,b}[(R_{a,b}(T_a) - K)^+ | \mathcal{F}_t]$$

and since we have just seen that $R$ is martingale under $\widehat{C}$, take (Jamshidian , Brigo 2003)

$$dR_{a,b}(t) = \sigma_{a,b} R_{a,b}(t) dW^{a,b}(t),$$

where $W^{a,b}$ is a Brownian motion under $\widehat{\mathbb{Q}}^{a,b}$, leading to a market formula

# CDS Options: Market Models (embedded Stochastic Intensity)

$$\mathsf{CallCDS}_{a,b}(t, K; \mathsf{L}_{\mathsf{GD}}) = \mathbb{E}\{1_{\{\tau > T_a\}} D(t, T_a) \bar{C}_{a,b}(T_a)(R_{a,b}(T_a) - K)^+ | \mathcal{G}_t\}$$
$$= 1_{\{\tau > t\}} \bar{C}_{a,b}(t)[R_{a,b}(t) N(d_1(t)) - K N(d_2(t))]$$
$$d_{1,2} = \left( \ln(R_{a,b}(t)/K) \pm (T_a - t)\sigma^2_{a,b}/2 \right) / (\sigma_{a,b}\sqrt{T_a - t}).$$

As happens in most markets, this Black-like formula could be used as a implied volatility quoting mechanism rather than as a real model formula.
Furthermore, the numeraire martingale framework is general and can include CDS Options smiles that may arise in the market in the coming years. Indeed, we are not forced to take $dR_{a,b}(t) = \sigma_{a,b} R_{a,b}(t) dW^{a,b}(t)$, but can also assume

$$dR^{PR}_{a,b}(t) = \nu_{a,b}(t, R^{PR}_{a,b}(t)) R^{PR}_{a,b}(t) dW^{a,b}(t)$$

with $\nu$ a suitable function of time and state. We might choose the CEV dynamics, a displaced diffusion dynamics, a mixture dynamics or an uncertain vol dynamics. But this is beyond the scope of this course...

# CDS Options and Callable Defaultable Floaters

Finally, given our observation on equivalence FRN/CDS, CDS options can be interpreted as optional components of single-date callable defaultable FRN. All solutions presented above thus apply to these products as well.

Now some examples of CDS implied volatilities (Model implementation by Marco Tarenghi).

C1 = Deutsche Telecom; C2 = Daimler Chrysler; C3 = France Telecom

Euro market; CDS options quotes as of March 26, 2004; $\text{Rec} = 0.4$; $\text{Lgd} = 1-0.4 = 0.6$; $T_0 =$ March 26 2004 $(0)$; We consider two possible maturities $T_a =$June 20 2004 (86d≈3m) and $T_a' =$Dec 20 2004 (269d≈9m); $T_b =$ june 20 2009 (5y87d); we consider receiver option quotes (puts on $R$) in basis points (i.e. 1E-4 units on a notional of 1). We obtain

# CDS Options and Callable Defaultable Floaters

| | Option: bid | mid | ask | $R_{0,b}(0)$ | $R^{PR}_{a,b}(0)$ | $R^{PR2}_{a,b}(0)$ | $K$ | $\sigma^{PR}_{a,b}$ | $\sigma^{PR(2)}_{a,b}$ |
|---|---|---|---|---|---|---|---|---|---|
| C1($T_a$) | 14 | 24 | 34 | 60 | 61.497 | 61.495 | 60 | 50.31 | 50.18 |
| C2 | 32 | 39 | 46 | 94.5 | 97.326 | 97.319 | 94 | 54.68 | 54.48 |
| C3 | 18 | 25 | 32 | 61 | 62.697 | 62.694 | 61 | 52.01 | 51.88 |
| C1($T'_a$) | 28 | 35 | 42 | 60 | 65.352 | 65.344 | 61 | 51.45 | 51.32 |

Implied volatilities are rather high when compared with typical interest-rate default free swaption volatilities. However, the values we find have the same order of magnitude as some of the values found by Hull and White (2003) via historical estimation.

Further, we see that while the option prices differ considerably, the related implied volatilities are rather similar. This shows the usefulness of a rigorous model for implied volatilities. The mere price quotes could have left one uncertain on whether the credit spread variabilities implicit in the different companies were quite different from each other or similar.

# CDS Options and Callable Defaultable Floaters

We analyze also the implied volatilities and CDS forward rates under different payoff formulations and under stress. The above Table shows the impact of changing postponement from PR to PR2 (PR2 is a second postponement possibility we did not consider in detail here). When we change formulation we maintain the same market $R_{0,b}(0)$'s and from them we re-strip the hazard functions needed to compute the forwards $R_{a,b}(0)$'s and the numeraire at time $0$. The change of postponement leaves both CDS forward rates and implied volatilities almost unchanged.

# CDS Options and Callable Defaultable Floaters

In the next Table we check the impact of the recovery rate on implied volatilities and CDS forward rates. Every time we change recovery we re-strip the hazard functions from the same market $R_{0,b}(0)$'s. We re-strip hazard functions because $R_{0,b}(0)$'s are given by the market and we cannot change them, whereas our uncertainty is on the recovery rate, that might change. As we can see from the table the impact of the recovery rate is rather small, but we have to keep in mind that the CDS option payoff is built in such a way that the recovery direct flow in LGD cancels and the recovery remains only implicitly inside the initial condition $R_{a,b}(0)$ for the dynamics of $R_{a,b}$. Notice indeed that REC does not appear explicitly in the payoff

$$1_{\{\tau > T_a\}} D(t, T_a) \bar{C}_{a,b}(T_a)(R_{a,b}(T_a) - K)^+$$

# CDS Options and Callable Defaultable Floaters

| | REC = 20% | REC = 30% | REC = 40% | REC = 50% | REC = 60% |
|---|---|---|---|---|---|
| $\sigma^{PR}_{a,b}$: | | | | | |
| C1($T_a$) | 50.02 | 50.14 | 50.31 | 50.54 | 50.90 |
| C2 | 54.22 | 54.42 | 54.68 | 55.05 | 55.62 |
| C3 | 51.71 | 51.83 | 52.01 | 52.25 | 52.61 |
| C1($T'_a$) | 51.13 | 51.27 | 51.45 | 51.71 | 52.10 |
| $R^{PR}_{a,b}$: | | | | | |
| C1($T_a$) | 61.488 | 61.492 | 61.497 | 61.504 | 61.514 |
| C2 | 97.303 | 97.313 | 97.326 | 97.346 | 97.374 |
| C3 | 62.687 | 62.691 | 62.697 | 62.704 | 62.716 |
| C1($T'_a$) | 65.320 | 65.334 | 65.352 | 65.377 | 65.415 |

Table 19: Impact of recovery rates on the implied volatility and on the CDS forward rates for the PR payoff. Vols are expressed as percentages and rates as basis points

# CDS Options and Callable Defaultable Floaters

In the next Table we check the impact of a shift in the simply compounded rates of the zero coupon interest rate curve on CDS forward rates and implied volatilities. Every time we shift the curve we recalibrate the hazard functions, while maintaining the same $R_{0,b}(0)$'s. We see that the shift has a more relevant impact than the recovery rate, an impact that remains small.

| | shift $-0.5\%$ | 0 | $+0.5\%$ |
|---|---|---|---|
| C1($T_a$) | 49.68 | 50.31 | 50.93 |
| C2 | 54.02 | 54.68 | 55.34 |
| C3 | 51.36 | 52.01 | 52.65 |

| shift $-0.5\%$ | 0 | $+0.5\%$ |
|---|---|---|
| 61.480 | 61.497 | 61.514 |
| 97.294 | 97.326 | 97.358 |
| 62.677 | 62.697 | 62.716 |

Table 20: Implied volatilities $\sigma_{a,b}$ (left, as percentages) and forward CDS rates $R^{PR}_{a,b}$ (right, as basis points) as the simply compounded rates are shifted uniformly for all maturities.

# Constant Maturity CDS with Market Models

**A "floating-rate" CDS.**

A contract that protects in $T_a, T_b$ can be in principle decomposed into a stream of contracts, each single contract protecting in $[T_{j-1}, T_j]$, for $j = a+1, \ldots, b$, say with protection payment LGD postponed to $T_j$ if default occurs in $[T_{j-1}, T_j]$.

In each period, the rate $R_j(T_{j-1})$ paid at $T_j$ makes the exchange fair, so that

"a contract offering protection LGD on a reference credit "C" in $[T_a, T_b]$ in exchange for payment of rates $R_{a+1}(T_a), \ldots, R_j(T_{j-1}), \ldots, R_b(T_{b-1})$ at times $T_{a+1}, \ldots, T_j, \ldots, T_b$ is fair"

i.e. has zero initial value. This product can be seen as a sort of floating rate CDS.

# Constant Maturity CDS with Market Models

**Constant Maturity CDS: Recall the definition**. Consider a contract protecting in $[T_a, T_b]$ against default of a reference credit "C".

If default occurs in $[T_a, T_b]$, a protection payment LGD is made from the protection seller "B" to the protection buyer "A" at the first $T_j$ following the default time.

This is called "protection leg".

In exchange for this protection "A" pays to "B" at each $T_j$ before default a "$c+1$–long" (constant maturity) CDS rate $R_{j-1,j+c}(T_{j-1})$ (times a year fraction $\alpha_j = T_j - T_{j-1}$), with "$c$" an integer larger than zero.

Notice that for $c = 0$ we would obtain the fair "floating rate" CDS above, whose initial value would be zero.

# Constant Maturity CDS

Standard CDS:

| Protection Seller B | $\rightarrow$ | protection LGD at default $\tau_C$ if $T_a < \tau_C \le T_b$ | $\rightarrow$ | Protection Buyer A |
|---|---|---|---|---|
| | $\leftarrow$ | rate $R$ at $T_{a+1}, \ldots, T_b$ or until default $\tau_C$ | $\leftarrow$ | |

"Floating Rate Equivalent" of Standard CDS:

| B | $\rightarrow$ | protection LGD at default $\tau_C$ if $T_a < \tau_C \le T_b$ | $\rightarrow$ | |
|---|---|---|---|---|
| | $\leftarrow$ | rate $R_{i-1,i}(T_{i-1})$ at $T_i = T_{a+1}, \ldots, T_b$ or until $\tau_C$ | $\leftarrow$ | A |

CMCDS:

| B | $\rightarrow$ | protection LGD at default $\tau_C$ if $T_a < \tau_C \le T_b$ | $\rightarrow$ | |
|---|---|---|---|---|
| | $\leftarrow$ | $R_{i-1,i+c}(T_{i-1})$ at $T_i = T_{a+1}, \ldots, T_b$ or until $\tau_C$ | $\leftarrow$ | A |

# Constant Maturity CDS (CMCDS) with Market Models: Main result

$$\mathsf{CDScm}_{a,b,c}(0, \mathsf{L}\text{GD}) = \sum_{j=a+1}^{b} \alpha_j \bar{P}(0, T_j) \left\{ \sum_{i=j}^{j+c} \frac{\alpha_i \bar{P}(0, T_i)}{\sum_{h=j}^{j+c} \alpha_h \bar{P}(0, T_h)} \cdot \right.$$

$$\left. \widetilde{R}_i(0) \exp\left[ T_{j-1}\sigma_i \cdot \left( \sum_{k=j+1}^{i} \rho_{j,k} \frac{\sigma_k \widetilde{R}_k(0)}{\widetilde{R}_k(0) + \mathsf{L}\text{GD}/\alpha_k} \right) \right] - R_j(0) \right\}$$

$R_k(0)$: one-period postponed P1 CDS forward rates for protection in $[T_{k-1}, T_k]$,

$$R_k(0) = \{R_{0,k}(0) \sum_{h=1}^{k} \alpha_h \bar{P}(0, T_h) - R_{0,k-1}(0) \sum_{h=1}^{k-1} \alpha_h \bar{P}(0, T_h)\} / \{\alpha_k \bar{P}(0, T_k)\}$$

$\widetilde{R}_k$ approx of the $R_k$ ($=$ in case of independence of interest rates and credit spreads)

$$R_k(0) \approx \widetilde{R}_k(0) = \mathsf{L}\text{GD}\{\bar{P}(0, T_{k-1}) P(0, T_k)/P(0, T_{k-1}) - \bar{P}(0, T_k)\} / \{\alpha_k \bar{P}(0, T_k)\}$$

$\sigma_k$ is the volatility of $R_k(t)$, constant (time-varying case can be easily generalized); $\rho_{i,j}$ inst correl $R_i$, $R_j$;

# Constant Maturity CDS (CMCDS) with Market Models: Main result

One-period CDS rates volatilities $\sigma_k$ can in principle be stripped from longer period CDS volatilities, similarly to how forward LIBOR rates volatilities can be stripped from swaptions volatilities. This is possible from an approximated volatility formula based on drift freezing (formula (6.58) for the LIBOR case in Brigo and Mercurio (2001)). Cascade methods are also available for this (as in Brigo and Morini (2004)), although for the time being the only available CDS options all have short maturities and the lack of a liquid market discourages this kind of approach.

The formula can be employed with stylized volatilities to have an idea of the impact of the "convexity adjustments". One may also consider historical volatilities and correlations.

As a further remark we notice that, if not for the exponential term (which vanishes for example when $\rho$'s are set to zero) this expression would be, not surprisingly,

$$\text{CDS}\text{CM}_{a,b,c}(0, \text{L}\text{GD}; \rho = 0) = \sum_{j=a+1}^{b} \alpha_j \bar{P}(0, T_j)(R_{j-1,j+c}(0) \; - R_{j-1,j}(0)) \tag{8}$$

Compare: The exponential term in the main formula is a sort of "convexity adjustment"

# CMCDS with Market Models: Formula derivation

$$R_j(t) := \text{L}_{\text{GD}} \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)} = \text{L}_{\text{GD}} \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)} =$$

$$= \text{L}_{\text{GD}} \frac{\mathbb{E}[D(t,T_{j-1})\mathbf{1}_{\{\tau>T_{j-1}\}} \boxed{D(t,T_j)/D(t,T_{j-1})} |\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)} = \ldots$$

$$\approx \text{L}_{\text{GD}} \frac{\mathbb{E}[D(t,T_{j-1})\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{F}_t] \boxed{P(t,T_j)/P(t,T_{j-1})} - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)}$$

$$= \text{L}_{\text{GD}} \frac{\bar{P}(t,T_{j-1})P(t,T_j)/P(t,T_{j-1}) - \bar{P}(t,T_j)}{\alpha_j \bar{P}(t,T_j)} = \frac{\text{L}_{\text{GD}}}{\alpha_j}\left(\frac{\bar{P}(t,T_{j-1})}{(1+\alpha_j F_j(t))\bar{P}(t,T_j)} - 1\right)$$

$$\approx \frac{\text{L}_{\text{GD}}}{\alpha_j}\left(\frac{\bar{P}(t,T_{j-1})}{(1+\alpha_j F_j(0))\bar{P}(t,T_j)} - 1\right) =: \widetilde{R}_j(t)$$

# CMCDS with Market Models: Formula derivation

$$\approx \frac{\text{LGD}}{\alpha_j}\left(\frac{\bar{P}(t,T_{j-1})}{(1+\alpha_j F_j(0))\bar{P}(t,T_j)} - 1\right) =: \widetilde{R}_j(t)$$

where $F$ is the forward LIBOR rate between $T_{j-1}$ and $T_j$. Notice:

$$\frac{\bar{P}(t,T_{j-1})}{\bar{P}(t,T_j)} = \left(\frac{\alpha_j}{\text{LGD}}\widetilde{R}_j + 1\right)(1+\alpha_j F_j(0)) > 1 \qquad (9)$$

as long as $\widetilde{R} > 0$, provided that $F_j(0) > 0$ as should be. This means that we are free to select any martingale dynamics for $\widetilde{R}_j$ under $\widehat{\mathbb{Q}}^{j-1,j}$, as long as $\widetilde{R}_j$ remains positive. Choose than such a family of $\widetilde{R}$ as building blocks

$$d\widetilde{R}_i(t) = \sigma_i(t)\widetilde{R}_i(t)dZ_i^i(t), \quad \text{for all } \; i$$

and define the $\bar{P}$ by using (9) to obtain inductively $\bar{P}(t,T_j)$ from $\bar{P}(t,T_{j-1})$ and from $\widetilde{R}_j$. **This way, the numeraires $\bar{P}$ become functions only of the $\widetilde{R}$'s, so that now the system is closed and all one has to model is the one-period rates $\widetilde{R}$ vector. No need to model two-period rates in this framework.**

# CMCDS with Market Models: Formula derivation

In this context the change of numeraire for diffusions

$$dZ^{Num2} = dZ^{Num1} - \rho \, \text{DiffusionCoefficient} \, (\ln(\text{Numeraire2}/\text{Numeraire1}))$$

becomes

$$dZ^j = dZ^i - \rho \text{DC} \left( \ln \left( \frac{\widehat{C}_{j-1,j}}{\widehat{C}_{i-1,i}} \right) \right)' dt = dZ^i - \rho \text{DC} \left( \ln \left( \frac{\bar{P}(t,T_j)}{\bar{P}(t,T_i)} \right) \right)' dt =$$

$$= dZ^i - \rho \text{DC} \ln \left[ \left( \prod_{h=j+1}^{i} \left( \frac{\alpha_h}{\text{L}_{\text{GD}}} \widetilde{R}_h + 1 \right) (1 + \alpha_h F_h(0)) \right) \right]' dt$$

$$= dZ^i - \rho \sum_{h=j+1}^{i} \text{DC} \ln \left( \left( \frac{\alpha_h}{\text{L}_{\text{GD}}} \widetilde{R}_h + 1 \right) (1 + \alpha_h F_h(0)) \right)' dt =$$

$$= dZ^i - \rho \sum_{h=j+1}^{i} \text{DC} \ln \left( \left( \frac{\alpha_h}{\text{L}_{\text{GD}}} \widetilde{R}_h + 1 \right) \right)' dt = dZ^i - \rho \sum_{h=j+1}^{i} \frac{1}{\widetilde{R}_h + \frac{\text{L}_{\text{GD}}}{\alpha_h}} \text{DC}(\widetilde{R}_h)' dt$$

# CMCDS with Market Models: Formula derivation

$$dZ_k^j = dZ_k^i - \sum_{h=j+1}^{i} \rho_{k,h} \frac{\sigma_h(t)\widetilde{R}_h}{\widetilde{R}_h + \frac{\mathsf{L}_{\mathsf{GD}}}{\alpha_h}} dt$$

from which we have the dynamics of $\widetilde{R}_i$ under $\mathbb{Q}^j$:

$$d\widetilde{R}_i = \sigma_i \widetilde{R}_i dZ_i^i = \sigma_i \widetilde{R}_i \left( dZ_i^j + \sum_{h=j+1}^{i} \rho_{j,h} \frac{\sigma_h \widetilde{R}_h}{\widetilde{R}_h + \frac{\mathsf{L}_{\mathsf{GD}}}{\alpha_h}} dt \right) =: \widetilde{R}_i(\tilde{\mu}_i^j(\widetilde{R})dt + \sigma_i dZ_i^j)$$

Consider the drift term. If we compute $E^{j-1,j}[\widetilde{R}_i(T_{j-1})]$ we obtain

$$\begin{aligned} E^{j-1,j}[\widetilde{R}_i(T_{j-1})] &\approx \widetilde{R}_i(0) \exp\left\{ \int_0^{T_{j-1}} \widetilde{\mu}_i^j(\widetilde{R}(0)) du \right\} \\ &= \widetilde{R}_i(0) \exp\left\{ \sum_{k=j+1}^{i} \frac{\widetilde{R}_k(0)}{\widetilde{R}_k(0) + \mathsf{L}_{\mathsf{GD}}/\alpha_k} \rho_{j,k} \int_0^{T_{j-1}} \sigma_i(u)\sigma_k(u) du \right\} \end{aligned}$$

# CMCDS with Market Models: Formula derivation

and, if we take volatilities $\sigma$ to be constant, we have

$$\approx \widetilde{R}_i(0) \exp\left\{ T_{j-1}\sigma_i \cdot \left( \sum_{k=j+1}^{i} \rho_{j,k} \frac{\sigma_k \widetilde{R}_k(0)}{\widetilde{R}_k(0) + \mathsf{L}\textsf{GD}/\alpha_k} \right) \right\}$$

Under independence between intensities and interest rates (and in particular under deterministic intensities, a common assumption when stripping one-period CDS rates from multi-period ones), by definition of $R_j$ it is easy to show that at time $0$,

$$R_j(0) = \widetilde{R}_j(0) = \mathsf{L}\textsf{GD}/\alpha_j \left( \frac{\mathbb{Q}(\tau > T_{j-1})}{\mathbb{Q}(\tau > T_j)} - 1 \right) \tag{10}$$

and hence

$$E^{j-1,j}[\widetilde{R}_i(T_{j-1})] = \frac{\mathsf{L}\textsf{GD}}{\alpha_i} \left( \frac{\mathbb{Q}(\tau > T_{i-1})}{\mathbb{Q}(\tau > T_i)} - 1 \right) \exp\left\{ T_{j-1}\sigma_i \sum_{k=j+1}^{i} \rho_{j,k}\sigma_k \left( 1 - \frac{\mathbb{Q}(\tau > T_k)}{\mathbb{Q}(\tau > T_{k-1})} \right) \right\}$$

The convexity effect vanishes if the ratio $\mathbb{Q}(\tau > T_j)/\mathbb{Q}(\tau > T_{j-1})$ is close to one.

# CMCDS with Market Models: Formula derivation

Now, based on the approximated dynamics and the related expectation above, we prove the main formula for CMCDS. We compute the price of the premium leg as

$$\sum_{j=a+1}^{b} \alpha_j \mathbb{E}_0[D(0,T_j)\mathbf{1}_{\{\tau>T_j\}} R_{j-1,j+c}(T_{j-1})] = \ldots$$

The first approx we consider comes from expressing multi-period $R$ as weighted averages of single period $R$, as we have seen when we introduced CDS forward rates in the beginning.

$$R_{j-1,j+c}(T_{j-1}) \approx \sum_{i=j}^{j+c} \bar{w}_i^j(0) R_i(T_{j-1}), \quad \bar{w}_i^j(0) = \frac{\alpha_i \bar{P}(0,T_i)}{\sum_{h=j}^{j+c} \alpha_h \bar{P}(0,T_h)}$$

Then by substituting this in the premium leg expression we have

## CMCDS with Market Models: Formula derivation

$$\ldots \approx \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{w}_i^j(0) \mathbb{E}_0[D(0,T_j) \mathbf{1}_{\{\tau > T_j\}} R_i(T_{j-1})] =$$

$$= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{w}_i^j(0) \mathbb{E}_0[D(0,T_j) R_i(T_{j-1}) \mathbb{E}(\mathbf{1}_{\{\tau > T_j\}} | \mathcal{F}_{T_j})]$$

$$= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \bar{w}_i^j(0) \mathbb{E}_0 \left[ \frac{B(0)}{B(T_j)} (R_i(T_{j-1}) \; \widehat{C}_{j-1,j}(T_j)) \right]$$

$$= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \bar{w}_i^j(0) \widehat{C}_{j-1,j}(0) \widehat{\mathbb{E}}_0^{j-1,j}[R_i(T_{j-1})] = \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{w}_i^j(0) \bar{P}(0,T_j) \widehat{\mathbb{E}}_0^{j-1,j}[R_i(T_{j-1})] = \ldots$$

where we have applied the change of numeraire, moving from the risk neutral numeraire $B$ to the numeraires $\widehat{C}_{j-1,j}$'s. The last expected value can be computed based on our earlier expression based on drift freezing, and we obtain the final formula.

## CMCDS with Market Models: A few numerical examples

We report input data and outputs for a name with relatively large CDS forward rates.

We consider the FIAT car company CDS market quotes as of December 20, 2004. Since we have seen earlier that under the SSRD model of Brigo and Alfonsi (2003) CDS prices depend very little on the correlation between interest rates and credit spreads, when stripping credit spreads from CDS data we may assume independence between interest rates and credit spreads.

This leads to the same technique we have seen earlier for the deterministic intensity model. Using this independence assumption, we strip default (or survival) probabilities from CDS quotes with increasing maturities.

# CMCDS with Market Models: A few numerical examples

We take as inputs the following Fiat CDS rates and use mid quotes

| $T_b$ | $R_{0,b}^{BID}(bps)$ | $R_{0,b}^{ASK}$ |
|---|---|---|
| 1Y | 99.9 | 175.57 |
| 2Y | 172.5 | 231.38 |
| 3Y | 243.73 | 286.13 |
| 5Y | 348.85 | 366.54 |
| 7Y | 380 | 410 |
| 10Y | 395.16 | 412.73 |

We take $\text{Rec} = 0.4$ (so that $\text{Lgd} = 0.6$). The input zero coupon curve, and the survival risk-neutral probabilities stripped from the above CDS quotes are reported below.

## CMCDS with Market Models: A few numerical examples

Outputs: We start by giving a table for

$$\text{Conv}(\sigma, \rho) := \text{CDS}\text{CM}_{a,b,c}(0, \text{L}\text{GD}, \sigma, \rho) - \text{CDS}\text{CM}_{a,b,c}(0, \text{L}\text{GD}; \rho = 0).$$

The first term is computed with our formula by assuming the volatilities $\sigma_i$ of forward one-period CDS rates $R_i$ to have a common value $\sigma$ and the pairwise correlations $\rho_{i,j}$ to have a common value $\rho$.

The second term is the simpler value where no correction due to CDS forward rate dynamics is accounted for. This difference then gives us the impact of volatilities and correlations of CDS rates on the CMCDS price. The difference is always positive, similarly to what happens to analogous constant maturity swaps in default free markets under similar conditions on volatilities and correlation. It is the impact of “convexity” on the CMCDS valuation.

# CMCDS with Market Models: A few numerical examples

We take $a = 0$, $b = 20$ (5y final maturity) and $c = 20$ (which means we are considering non-standard 5y3m CDS rates in the CMCDS premium leg, $c = 19$ would amount to a 5y CDS rate).

We obtain

| Conv$(\sigma, \rho)$ | $\rho$: 0.7 | 0.8 | 0.9 | 0.99 |
|---|---|---|---|---|
| $\sigma$: 0.1 | 0.000659 | 0.000754 | 0.000848 | 0.000933 |
| 0.2 | 0.002662 | 0.003047 | 0.003435 | 0.003784 |
| 0.4 | 0.011066 | 0.012742 | 0.014442 | 0.015995 |
| 0.6 | 0.026619 | 0.030964 | 0.035464 | 0.039652 |

The "convexity difference" increases with both correlation and volatility, as expected.

# CMCDS with Market Models: A few numerical examples

The next table reports the so called “participation rate” $\phi_{a,b,c}(\sigma,\rho)$ for a CMCDS with final $T_b = 5y$ ($a = 0, b = 20$, recalling that resets occur quarterly), with $5y3m$ constant maturity CDS rates ($c = 20$),

$$\phi_{0,20,20}(\sigma,\rho) = \frac{\text{“premium leg CDS”}}{\text{“premium leg CMCDS”}} = \frac{\sum_{j=1}^{20} \alpha_j \bar{P}(0,T_j) R_{0,20}(0)}{\sum_{j=1}^{20} \alpha_j \mathbb{E}_0^{j-1,j}[D(0,T_j)\mathbf{1}_{\{\tau>T_j\}} R_{j-1,j+20}(T_{j-1})]},$$

The CMCDS premium leg is computed with our approximated market model based on one-period rates $\widetilde{R}$. As we see from the outputs, the participation rate increases with volatility and correlation, as is expected from the “convexity adjustment” effect.

| $\phi_{0,20,20}(\sigma,\rho)$ | $\rho$: 0.7 | 0.8 | 0.9 | 0.99 |
|---|---|---|---|---|
| $\sigma$: 0.1 | 0.71358 | 0.71325 | 0.71292 | 0.71262 |
| 0.2 | 0.70664 | 0.70532 | 0.704 | 0.70281 |
| 0.4 | 0.67894 | 0.67368 | 0.66842 | 0.66368 |
| 0.6 | 0.63302 | 0.62128 | 0.60957 | 0.59907 |

# CMCDS with Market Models: A few numerical examples

Finally, we fix volatilities and correlations and check how the patterns change when changing final maturity $T_b = T_i$. We consider time 0 and $T_a = 0$:

$$x_i = \frac{\text{“Constant maturity rate”}}{\text{“standard rate”}} = \frac{R_{i-1,i+c}(0)}{R_{0,b}(0)}, \quad i = 1, \ldots, b$$

$$y_i = \frac{\mathbb{E}_0^{i-1,i}[D(0,T_i)\mathbf{1}_{\{\tau>T_i\}} R_{i-1,i+c}(T_{i-1})]}{\bar{P}(0,T_i) R_{0,b}(0)}, \quad i = 1, \ldots, b$$

$$z_i = \frac{\mathbb{E}_0^{i-1,i}[D(0,T_i)\mathbf{1}_{\{\tau>T_i\}} R_{i-1,i+c}(T_{i-1})]}{\bar{P}(0,T_i) R_{i-1,i+c}(0)}, \quad i = 1, \ldots, b$$

$$\psi_i = \frac{\text{“premium leg CDS”}}{\text{“premium leg CMCDS”}} = \frac{\sum_{j=1}^{i} \alpha_j \bar{P}(0,T_j) R_{0,i}(0)}{\sum_{j=1}^{i} \alpha_j \bar{P}(0,T_j) R_{j-1,j+c}(0)}, \; i = 1, \ldots, b$$

$$\phi_i = \frac{\text{“premium leg CDS”}}{\text{“premium leg CMCDS with convexity”}} = \frac{\sum_{j=1}^{i} \alpha_j \bar{P}(0,T_j) R_{0,i}(0)}{\sum_{j=1}^{i} \alpha_j \mathbb{E}_0^{j-1,j}[D(0,T_j)\mathbf{1}_{\{\tau>T_j\}} R_{j-1,j+c}(T_{j-1})]}.$$

| $x_i$ | $y_i$ | $z_i$ | $\psi_i$ | $\phi_i$ | |
|---|---|---|---|---|---|
| | | | | | $\sigma$ = 0.4; |
| | | | | | $\rho$ = 0.9; |
| 1.0668 | 1.0668 | 1 | 0.37773 | 0.37773 | |
| 1.1288 | 1.1359 | 1.0063 | 0.36281 | 0.36162 | REC = 0.4; |
| 1.1914 | 1.2075 | 1.0135 | 0.35281 | 0.35039 | a=0; |
| 1.2525 | 1.2792 | 1.0214 | 0.34359 | 0.33993 | c = 20; |
| 1.3107 | 1.3495 | 1.0297 | 0.33512 | 0.33024 | b = 20; |
| 1.3673 | 1.4193 | 1.038 | 0.34187 | 0.33548 | |
| 1.4171 | 1.4826 | 1.0462 | 0.36905 | 0.36064 | |
| 1.4515 | 1.53 | 1.0541 | 0.40755 | 0.39664 | |
| 1.4716 | 1.5622 | 1.0616 | 0.45262 | 0.43881 | |
| 1.4798 | 1.5818 | 1.0689 | 0.49477 | 0.47785 | |
| 1.4837 | 1.5979 | 1.0769 | 0.52661 | 0.50671 | |
| 1.4905 | 1.6175 | 1.0852 | 0.55072 | 0.52799 | |
| 1.4999 | 1.6403 | 1.0936 | 0.56931 | 0.54384 | |
| 1.5122 | 1.666 | 1.1018 | 0.58674 | 0.55846 | |
| 1.5236 | 1.69 | 1.1092 | 0.60704 | 0.57574 | |
| 1.5275 | 1.706 | 1.1168 | 0.62715 | 0.5928 | |
| 1.5274 | 1.7174 | 1.1244 | 0.64681 | 0.60938 | |
| 1.5249 | 1.7236 | 1.1303 | 0.67017 | 0.62939 | |
| 1.5106 | 1.7173 | 1.1368 | 0.69254 | 0.64843 | |
| 1.4924 | 1.7047 | 1.1422 | 0.71589 | 0.66842 | |

# CMCDS with Market Models: A few numerical examples

The $x_i$'s measure how the constant maturity CDS rate differs multiplicatively from the standard CDS rate. We find an increasing pattern in $T_i$ as partly expected from the fact that the input CDS rates $R_{0,b}^{BID,ASK}$ are increasing with respect to maturity $T_b$.

The $y_i$'s measure the same effect while taking into account "convexity", i.e. future randomness of the payoff and correlation. The $y_i$'s would reduce to the $x_i$'s if correlations $\rho$ were taken equal to $0$. The $y$ maintain the increasing pattern.

The $z_i$'s measure the multiplicative impact of "convexity", in that they are due to contributions stemming from volatilities $\sigma$ and correlations $\rho$ of CDS rates. The impact is increasing with maturity $T_i$, as expected from the sign in the exponent of the convexity adjustments and from the positive signs of correlations (and volatilities).

# CMCDS with Market Models: A few numerical examples

Finally, as seen above, the $\psi_i$'s are the so called "participation rates" for different terminal maturities $T_i$. They give the ratio between the premium leg in a standard CDS protecting in $[0, T_i]$ and the premium leg in CMCDS for the same protection interval when ignoring the convexity adjustment due to correlation and volatilities. The $\phi_i$'s are the participation rates computed when taking into account convexity due to volatilities and correlations.

In the table above for $\phi_i$ we obtain an initially decreasing pattern followed by a longer increasing pattern for both $\psi$ and $\phi$. Notice that, on the longest participation rate, in the last row of the related table, convexity has an impact moving from a $71.59\%$ participation rate when not including "convexity" to a $66.84\%$ participation rate when including convexity. There is a $4.8\%$ difference in the participation rate of this FIAT 5y-5y3m CMCDS with correlations set at 0.9 and volatilities at $40\%$.

## Single Name Models: Reduced Form. Summary

We have seen:

- Explicit Reduced Form (intensity) Models;
  - Constant intensity, standard Poisson process
  - Time varying deterministic intensity, time inhomogeneous Poisson Process, Credit Spread modeling; Piecewise constant and piecewise linear intensities, CDS calibration;
  - Time varying Stochastic intensity, Cox processes, Credit spreads and their volatilities; SSRD CIR++ model; CDS calibration, interest rate calibration. CDS options formula;
- Market Models; CDS options implied volatilty; CDS options smile; Constant Maturity CDS with convexity adjustment.