

CENTRE FOR COMPUTATIONAL FINANCE AND ECONOMICS AGENTS

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CF907-7-SP: Fixed-Income Asset Pricing, Default Risk, and Credit Ratings

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Credit and Default Modeling

Unit 1: SINGLE NAME CREDIT DERIVATIVES

Unit 2: SINGLE NAME MODELS: REDUCED FORM

Unit 3: SINGLE NAME MODELS: STRUCTURAL

Unit 4: MULTI NAME CREDIT DERIVATIVES

Unit 5: MULTI NAME REDUCED FORM MODELS AND COPULAS

Unit 6: IMPLIED ETL and DYNAMIC LOSS MODELS

Unit 7: COUNTERPARTY RISK VALUATION

Credit and Default Modeling

UNIT 1

SINGLE NAME CREDIT DERIVATIVES

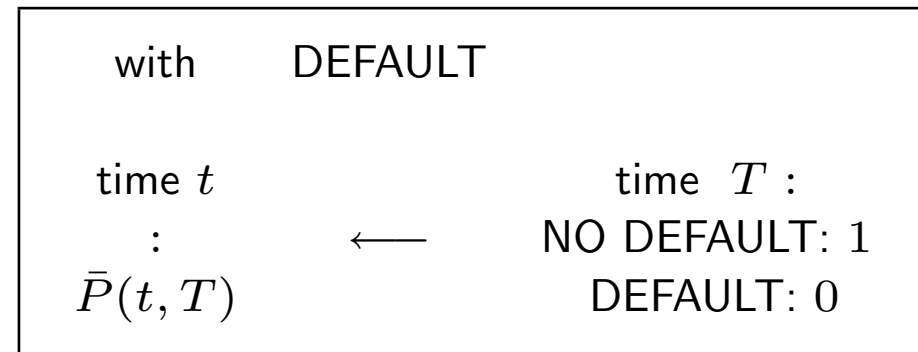
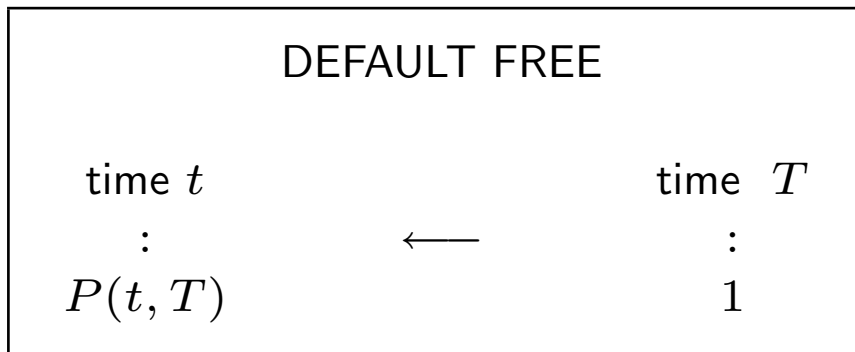
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UNIT 1: SINGLE NAME CREDIT DERIVATIVES

- Defaultable (corporate) zero coupon bonds;
- Defaultable coupon bonds;
- Defaultable floaters;
- Credit Default Swaps (CDS);
- CDS rates;
- CDS and Floaters;
- Constant Maturity CDS
- Equity payoffs with counterparty risk;
- Equity Return Swaps with counterparty risk;
- Equity Default Swaps;

Defaultable (corporate) zero coupon bonds

Similarly to the zero coupon bond $P(t, T)$ being one of the possible fundamental quantities for describing the interest-rate curve, we now consider a defaultable bond $\bar{P}(t, T)$ as a possible fundamental variable for describing the defaultable market.



When considering default, we have a random time τ representing the time at which a given company defaults.

τ : Default time of the company

Defaultable (corporate) zero coupon bonds

The value of a bond issued by the company and promising the payment of 1 at time T , as seen from time t , is the risk neutral expectation of the discounted payoff

BondPrice = Expectation[Discount \times Payoff]

$$P(t, T) = \mathbb{E}\{D(t, T) \mathbf{1} | \mathcal{G}_t\}, \quad \mathbf{1}_{\{\tau > t\}} \bar{P}(t, T) := \mathbb{E}\{D(t, T) \mathbf{1}_{\{\tau > T\}} | \mathcal{G}_t\}$$

where \mathcal{G}_t represents the flow of information on whether default occurred before t and if so at what time exactly, and on the default free market variables up to t . D is the stochastic discount factor between two dates, depending on interest rates, and represents discounting.

The “indicator” function $\mathbf{1}_{\text{condition}}$ is 1 if “condition” is satisfied and 0 otherwise. In particular, $\mathbf{1}_{\{\tau > T\}}$ reads 1 if default τ did not occur before T , and 0 in the other case.

We understand then that (ignoring recovery) $\mathbf{1}_{\{\tau > T\}}$ is the correct payoff for a corporate bond at time T : the contract pays 1 if the company has not defaulted, and 0 if it defaulted before T , according to our earlier stylized description.

Defaultable (corporate) zero coupon bonds

If we include a recovery amount REC to be paid at default τ in case of early default, we have as discounted payoff at time t

$$D(t, T)\mathbf{1}_{\{\tau > T\}} + \text{REC}D(t, \tau)\mathbf{1}_{\{\tau \leq T\}}$$

If we include a recovery amount REC paid at maturity T , we have as discounted payoff

$$D(t, T)\mathbf{1}_{\{\tau > T\}} + \text{REC}D(t, T)\mathbf{1}_{\{\tau \leq T\}}$$

Defaultable (corporate) coupon bonds

Bond issued by counterparty “C” with default time τ paying coupons $c_{a+1}, c_{a+2}, \dots, c_b$ at times T_{a+1}, \dots, T_b , with recovery amount REC: Discounted payoff at time t is (assuming a constant absolute recovery cash flow “REC” at the first T_i following default):

Coupons if no default + Notional reimbursement if no default + recovery if early default

$$\sum_{i=a+1}^b c_i D(t, T_i) \mathbf{1}_{\{\tau > T_i\}} + D(t, T_b) \mathbf{1}_{\{\tau > T_b\}} + \text{REC} \sum_{i=a+1}^b D(t, T_i) \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}}$$

Defaultable (corporate) Floating rate note (Floater)

With Default, the floating note disc payoff, subtracting the notional invested in T_a , is

$$\Pi_{\text{DFRN}_{a,b}} = -D(t, T_a)\mathbf{1}_{\{\tau > T_a\}} + \sum_{i=a+1}^b \alpha_i D(t, T_i)(L(T_{i-1}, T_i) + X)\mathbf{1}_{\{\tau > T_i\}}$$

notional investment at T_a + LIBOR + spread payments in T_{a+1}, \dots, T_b
if no early default at dates T_i preceding default τ

$$+ D(t, T_b)\mathbf{1}_{\{\tau > T_b\}} + \text{REC} \sum_{i=a+1}^b D(t, T_i)\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}}$$

+final notional if no default +recovery in case of early default

$L(S, T)$ is the LIBOR rate resetting at S for maturity T , whereas X is the spread.

Fundamental Credit Derivatives: Credit Default Swaps

“It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts.”

Sherlock Holmes, *A Scandal in Bohemia*, quoted by KMV's J.R. Bohn on a credit risk survey paper.

Credit Default Swaps are basic protection contracts that became quite liquid in the last few years. CDS's are now actively traded and have become a sort of basic product of the credit derivatives area, analogously to interest-rate swaps and FRA's being basic products in the interest-rate derivatives world.

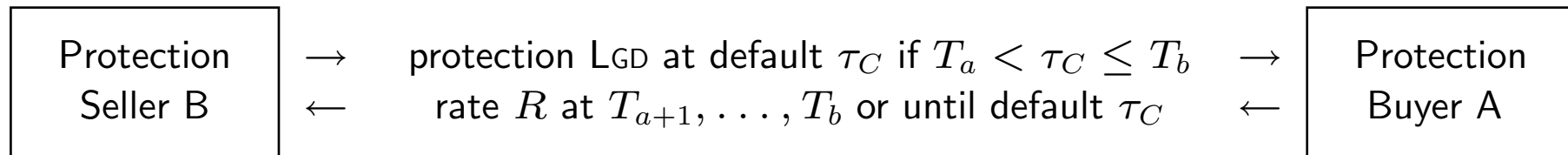
As a consequence, the need is no longer to have a model to be used to value CDS's, but rather to consider a model that can be *calibrated* to CDS's, i.e. to take CDS's as model inputs (rather than outputs), in order to price more complex credit derivatives.

CDS options are not liquid yet, but the interest for these products is growing in the market. We may expect models will have to incorporate CDS options rather than price them in a near future, similarly to what happened to CDS themselves.

Fundamental Credit Derivatives: CDS's

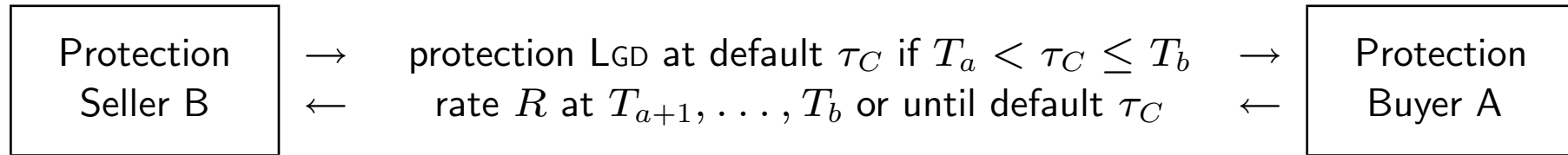
A CDS contract ensures protection against default. Two companies "A" (Protection buyer) and "B" (Protection seller) agree on the following.

If a third company "C" (Reference Credit) defaults at time τ , with $T_a < \tau < T_b$, "B" pays to "A" a certain (deterministic) cash amount LGD. In turn, "A" pays to "B" a rate R at times T_{a+1}, \dots, T_b or until default. Set $\alpha_i = T_i - T_{i-1}$ and $T_0 = 0$.



(protection leg and premium leg respectively). The cash amount LGD is a *protection* for "A" in case "C" defaults. Typically $\text{LGD} = \text{notional}$, or "notional - recovery" = $1 - \text{REC}$. A typical stylized case occurs when "A" has bought a corporate bond issued by "C" and is waiting for the coupons and final notional payment from "C": If "C" defaults before the corporate bond maturity, "A" does not receive such payments. "A" then goes to "B" and buys some protection against this risk, asking "B" a payment that roughly amounts to the loss on the bond (e.g. notional minus deterministic recovery) that A would face in case "C" defaults.

Fundamental Credit Derivatives: CDS's



Usually, at evaluation time (t) the amount $R = R_{a,b}(t)$ is set at a value that makes the contract fair, i.e. such that the present value of the two exchanged flows is zero. This is how the market quotes CDS's: CDS are quoted via their fair R 's (Bid and Ask). Formally we may write the (Running) CDS discounted value to "B" at time $t < T_a$ as $\Pi_{\text{RCDS}_{a,b}}(t) :=$

$$D(t, \tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^b D(t, T_i)\alpha_i R\mathbf{1}_{\{\tau > T_i\}} - \mathbf{1}_{\{T_a < \tau \leq T_b\}}D(t, \tau) \text{ LGD}$$

Accrued rate at default + CDS Rate payments if no default - Protection paym at default

where $T_{\beta(\tau)}$ is the first of the T_i 's following τ .

Fundamental Credit Derivatives. CDS's: Postponed Payoff

$$\Pi_{\text{RCDS}_{a,b}}(t) = D(t, \tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^b D(t, T_i)\alpha_i R\mathbf{1}_{\{\tau > T_i\}} - \mathbf{1}_{\{T_a < \tau \leq T_b\}} D(t, \tau) \text{LGD}$$

Accrued rate at default + CDS Rate payments if no default - Protection paym at default

Consider instead the two POSTPONED PAYMENT alternatives:

1) If default τ arrives in the interval $(T_{i-1}, T_i]$, move it to T_i and set the accrual period for the accrual term in rate R to zero;

2) If default τ arrives in the interval $(T_{i-1}, T_i]$, move it to T_i and set the accrual period for the accrual term in rate R to the whole $(T_{i-1}, T_i]$;

The difference between the two postponement strategies is whether to add a R payment for the date that has been postponed or not. For small ϵ , in paths where $\tau = T_j + \epsilon$ the first postponed payoff is a better approximation, whereas in paths where $\tau = T_j - \epsilon$ the second one is better.

Fundamental Credit Derivatives. CDS's: Postponed Payoff

The postponed payoffs read

$$\begin{aligned}\Pi_{\text{PRCDS}_{a,b}}(t) &= \sum_{i=a+1}^b D(t, T_i) \alpha_i R \mathbf{1}_{\{\tau \geq T_i\}} - \sum_{i=a+1}^b D(t, T_i) \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}} \text{LGD} \\ \Pi_{\text{PR2CDS}_{a,b}}(t) &= \sum_{i=a+1}^b D(t, T_i) \alpha_i R \mathbf{1}_{\{\tau > T_{i-1}\}} - \sum_{i=a+1}^b D(t, T_i) \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}} \text{LGD}\end{aligned}$$

CDS Rate payments if no default - Protection paym at first T_i after default

Note that $\mathbf{1}_{\{\tau > T_{i-1}\}} = \mathbf{1}_{\{\tau > T_i\}} + \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}}$ so that the two postponed payoffs differ of the term

$$\Pi_{\text{PR2CDS}_{a,b}}(t) - \Pi_{\text{PRCDS}_{a,b}}(t) = \sum_{i=a+1}^b D(t, T_i) \alpha_i R \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}}$$

which is exactly the postponement of the accrual term $D(t, \tau)(\tau - T_{\beta(\tau)-1}) R \mathbf{1}_{\{T_a < \tau < T_b\}}$

Fundamental Credit Derivatives. CDS's: Risk Neutral Valuation

Denote by $CDS_{a,b}(t, R, LGD)$, $PRCDS_{a,b}(t, R, LGD)$, $PR2CDS_{a,b}(t, R, LGD)$

the time t *price* of the above Running standard and postponed CDS's *payoffs*.

As usual, the price associated to a discounted payoff is its *risk neutral expectation*.

The resulting pricing formula depends on the assumptions on interest-rate dynamics and on the default time τ (reduced form models, structural models...).

Fundamental Credit Derivatives. CDS's: Risk Neutral Valuation

In general by risk-neutral valuation we can compute the CDS price at time 0 (or at any other time similarly):

$$\text{CDS}_{a,b}(0, R, \text{LGD}) = \mathbb{E}\{\Pi_{\text{RCDS}_{a,b}}(0)\},$$

$$\text{PRCDS}_{a,b}(0, R, \text{LGD}) = \mathbb{E}\{\Pi_{\text{PRCDS}_{a,b}}(0)\},$$

$$\text{PR2CDS}_{a,b}(0, R, \text{LGD}) = \mathbb{E}\{\Pi_{\text{PR2CDS}_{a,b}}(0)\}$$

with the CDS discounted payoffs defined earlier. \mathbb{E} denotes the risk-neutral expectation, the related measure being denoted by \mathbb{Q} .

However, we will not use the formulas resulting from this approach to price CDS that are already quoted in the market. Rather, we will invert these formulas in correspondence of market CDS quotes to calibrate our models to the CDS quotes themselves. We will give examples of this later.

Now let us have a look at some particular formulas resulting from the general risk neutral approach through some simplifying assumptions.

Default independent of interest rates: CDS Model-independent formulas

Assume the stochastic discount factors $D(s, t)$ to be independent of the default time τ for all possible $0 < s < t$. The premium leg of the CDS at time 0 is:

$$\begin{aligned}
 \text{PremiumLeg}_{a,b}(R) &= \mathbb{E}[D(0, \tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a < \tau < T_b\}}] + \\
 &\quad + \sum_{i=a+1}^b \mathbb{E}[D(0, T_i)\alpha_i R\mathbf{1}_{\{\tau \geq T_i\}}] \\
 &= \mathbb{E} \left[\int_{t=0}^{\infty} D(0, t)(t - T_{\beta(t)-1})R\mathbf{1}_{\{T_a < t < T_b\}}\mathbf{1}_{\{\tau \in [t, t+dt]\}} \right] + \\
 &\quad + \sum_{i=a+1}^b \mathbb{E}[D(0, T_i)]\alpha_i R \mathbb{E}[\mathbf{1}_{\{\tau \geq T_i\}}] =
 \end{aligned}$$

$$\begin{aligned}
\text{PremiumLeg}_{a,b}(R) &= \int_{t=T_a}^{T_b} \mathbb{E}[D(0, t)(t - T_{\beta(t)-1})R \mathbf{1}_{\{\tau \in [t, t+dt)\}}] + \\
&\quad + \sum_{i=a+1}^b P(0, T_i)\alpha_i R \mathbb{Q}(\tau \geq T_i) \\
&= \int_{t=T_a}^{T_b} \mathbb{E}[D(0, t)](t - T_{\beta(t)-1})R \mathbb{E}[\mathbf{1}_{\{\tau \in [t, t+dt)\}}] + \\
&\quad + \sum_{i=a+1}^b P(0, T_i)\alpha_i R \mathbb{Q}(\tau \geq T_i) \\
&= R \int_{t=T_a}^{T_b} P(0, t)(t - T_{\beta(t)-1})\mathbb{Q}(\tau \in [t, t + dt)) + \\
&\quad + R \sum_{i=a+1}^b P(0, T_i)\alpha_i \mathbb{Q}(\tau \geq T_i),
\end{aligned}$$

where we have used independence in factoring the above expectations.

Default independent of interest rates: CDS Model-independent formulas

We have thus, by rearranging terms and introducing a “unit-premium” premium leg (sometimes called “DV01”, “Risky duration” or “annuity”):

$$\begin{aligned}
 \text{PremiumLeg}_{a,b}(R; P(0, \cdot), \mathbb{Q}(\tau > \cdot)) &= R \text{PremiumLeg1}_{a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)), \\
 \text{PremiumLeg1}_{a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)) &:= - \int_{T_a}^{T_b} P(0, t)(t - T_{\beta(t)-1}) dt \boxed{\mathbb{Q}(\tau \geq t)} \\
 &\quad + \sum_{i=a+1}^b P(0, T_i) \alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)} \quad (1)
 \end{aligned}$$

This model independent formula uses the initial market zero coupon curve (bonds) at time 0 (i.e. $P(0, \cdot)$) and the survival probabilities $\mathbb{Q}(\tau \geq \cdot)$ at time 0 (terms in the boxes).

A similar formula holds for the protection leg, again under independence between default τ and interest rates.

Default independent of interest rates: CDS Model-independent formulas

$$\begin{aligned}
 \text{ ProtecLeg}_{a,b}(\text{LGD}) &= \mathbb{E}[\mathbf{1}_{\{T_a < \tau \leq T_b\}} D(0, \tau) \text{LGD}] \\
 &= \text{LGD} \mathbb{E} \left[\int_{t=0}^{\infty} \mathbf{1}_{\{T_a < t \leq T_b\}} D(0, t) \mathbf{1}_{\{\tau \in [t, t+dt)\}} \right] = \text{LGD} \left[\int_{t=T_a}^{T_b} \mathbb{E}[D(0, t) \mathbf{1}_{\{\tau \in [t, t+dt)\}}] \right] \\
 &= \text{LGD} \int_{t=T_a}^{T_b} \mathbb{E}[D(0, t)] \mathbb{E}[\mathbf{1}_{\{\tau \in [t, t+dt)\}}] = \text{LGD} \int_{t=T_a}^{T_b} P(0, t) \mathbb{Q}(\tau \in [t, t + dt))
 \end{aligned}$$

so that we have, by introducing a “unit-notional” protection leg:

$$\text{ ProtecLeg}_{a,b}(\text{LGD}; P(0, \cdot), \mathbb{Q}(\tau > \cdot)) = \text{LGD} \text{ ProtecLeg}_{1,a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)),$$

$$\text{ ProtecLeg}_{1,a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)) := - \int_{T_a}^{T_b} P(0, t) dt \boxed{\mathbb{Q}(\tau \geq t)} \quad (2)$$

Default independent of interest rates: CDS Model-independent formulas

$$\text{ ProtecLeg}_{a,b}(\text{LGD}; P(0, \cdot), \mathbb{Q}(\tau > \cdot)) = \text{LGD} \text{ ProtecLeg}_{1,a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)),$$

$$\text{ ProtecLeg}_{1,a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)) := - \int_{T_a}^{T_b} P(0, t) d_t \boxed{\mathbb{Q}(\tau \geq t)}$$

This formula too is model independent given the initial zero coupon curve (bonds) at time 0 observed in the market and given the survival probabilities at time 0 (term in the box).

The total (Receiver) CDS price can be written as

$$\text{CDS}_{a,b}(t, R, \text{LGD}; \mathbb{Q}(\tau > \cdot)) = R \text{PremiumLeg}_{1,a,b}(\mathbb{Q}(\tau > \cdot)) - \text{LGD} \text{ ProtecLeg}_{1,a,b}(\mathbb{Q}(\tau > \cdot))$$

$$= R \left[- \int_{T_a}^{T_b} P(0, t)(t - T_{\beta(t)-1}) d_t \boxed{\mathbb{Q}(\tau \geq t)} + \sum_{i=a+1}^b P(0, T_i) \alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)} \right] + \\ + \text{LGD} \left[\int_{T_a}^{T_b} P(0, t) d_t \boxed{\mathbb{Q}(\tau \geq t)} \right]$$

Default independent of interest rates: CDS Model-independent formulas

$$\text{CDS}_{a,b}(t, R, \text{LGD}; \mathbb{Q}(\tau > \cdot)) = \text{LGD} \left[\int_{T_a}^{T_b} P(0, t) d_t \boxed{\mathbb{Q}(\tau \geq t)} \right] +$$

$$R \left[- \int_{T_a}^{T_b} P(0, t)(t - T_{\beta(t)-1}) d_t \boxed{\mathbb{Q}(\tau \geq t)} + \sum_{i=a+1}^b P(0, T_i) \alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)} \right]$$

The integrals in the survival probabilities given in the above formulas can be valued as Stieltjes integrals in the survival probabilities themselves, and can easily be approximated numerically by summations through Riemann-Stieltjes sums, considering a low enough discretization time step.

Default independent of interest rates: CDS Model-independent formulas

The market quotes, at time 0, the fair $R = R_{0,b}^{\text{mktMID}}(0)$ (actually bid and ask quotes are available for this fair R) equating the two legs for a set of CDS with initial protection time $T_a = 0$ and final protection time $T_b \in \{1y, 2y, 3y, 4y, 5y, 6y, 7y, 8y, 9y, 10y\}$, although often only a subset of the maturities $\{1y, 3y, 5y, 7y, 10y\}$ is available. Solve then

$$\text{CDS}_{0,b}(t, R_{0,b}^{\text{mktMID}}(0), \text{LGD}; \mathbb{Q}(\tau > \cdot)) = 0$$

in portions of $\mathbb{Q}(\tau > \cdot)$ starting from $T_b = 1y$, finding the market implied survival $\{\mathbb{Q}(\tau \geq t), t \leq 1y\}$; plugging this into the $T_b = 2y$ CDS legs formulas, and then solving the same equation with $T_b = 2y$, we find the market implied survival $\{\mathbb{Q}(\tau \geq t), t \in (1y, 2y]\}$, and so on up to $T_b = 10y$.

This is a way to strip survival probabilities from CDS quotes in a model independent way. No need to assume an intensity or a structural model for default here.

Default independent of interest rates: CDS Model-independent formulas

However, the market in doing the above stripping typically resorts to hazard functions, assuming existence of hazard functions associated with the default time. We now assume existence of a deterministic intensity, as in deterministic intensity models, and briefly consider the notion of implied deterministic cumulated intensity (hazard function). **If the intensity γ is taken CONSTANT, implying**

$$\mathbb{Q}(\tau > t) = e^{-\gamma t}, \quad \mathbb{Q}(\tau \in [t, t + dt) | \tau \geq t) = \gamma dt,$$

and **if we assume the premium leg of the CDS pays continuously** the rate R rather than at discrete times T_i , the CDS price becomes

Default independent of interest rates: CDS Model-independent formulas

$$\text{CDS}_{a,b}(t, R, \text{LGD}; \mathbb{Q}(\tau > \cdot)) = -\text{LGD} \left[\int_{T_a}^{T_b} P(0, t) \gamma \mathbb{Q}(\tau \geq t) dt \right] + R \left[\int_{T_a}^{T_b} P(0, t) \mathbb{Q}(\tau \geq t) dt \right]$$

If we take $a = 0$ and insert the market CDS rate $R = R_{0,b}^{\text{mkt MID}}(0)$ in the premium leg the CDS present value should be zero. Solve $\text{CDS}_{a,b}(t, R, \text{LGD}; \mathbb{Q}(\tau > \cdot)) = 0$,

$$\gamma = \gamma_{0,b} = \frac{R_{0,b}^{\text{mkt MID}}(0)}{\text{LGD}}$$

from which we see that **the CDS rate is indeed a sort of CREDIT SPREAD, or INTENSITY**. In general, however, premium payments are not continuous and the intensity is taken to be time-varying, leading to

CDS valuation with determ intensities: Implied intensity γ and Γ

$$\begin{aligned} \text{CDS}_{a,b}(t, R, \text{LGD}; \Gamma(\cdot)) = & \mathbf{1}_{\{\tau > t\}} \left[R \int_{T_a}^{T_b} P(t, u) (T_{\beta(u)-1} - u) d(e^{-(\Gamma(u)-\Gamma(t))}) \right. \\ & \left. + \sum_{i=a+1}^b P(t, T_i) R \alpha_i e^{-(\Gamma(T_i)-\Gamma(t))} + \text{LGD} \int_{T_a}^{T_b} P(t, u) d(e^{-(\Gamma(u)-\Gamma(t))}) \right] \end{aligned}$$

This holds when τ is first jump time of a **time-inhomogeneous Poisson Process** with intensity γ and hazard function $\Gamma(t) = \int_0^t \gamma(u) du$. (more on this later). Market Quoting Mechanism: **The market quotes fair R^{mkt} 's that make the CDS value equal to zero.** One may wish to **calibrate** the model's Γ to such prices to value different payoffs. Find the Γ^{mkt} 's solving

$$\text{CDS}_{0,b}(0, R_{0,b}^{\text{mkt MID}}(0), \text{LGD}; \Gamma^{\text{mkt}}(0 \div T_b)) = 0, \quad T_b = 1y, 2y, 3y, 5y, 7y, 10y$$

If we are given $R_{0,b}^{\text{mkt MID}}(0)$ for different maturities T_b , we can assume a *piecewise linear* (or at times constant) γ , and invert prices in an iterative way as T_b increases, deriving each time the new part of γ that is consistent with the R for the new increased maturity.

CDS's with deterministic intensities: A first Example

September 10th, 2003 Recovery Rate = 40%

Maturity T_b (yr)	Maturity (dates)	$R_{0,b}$ (bps)
1	20-Sep-04	192.5
3	20-Sep-06	215
5	20-Sep-08	225
7	20-Sep-10	235
10	20-Sep-13	235

Table 1: Maturity dates and corresponding Parmalat CDS quotes in bps for September 10th, 2003.

CDS's with deterministic intensities: A first Example

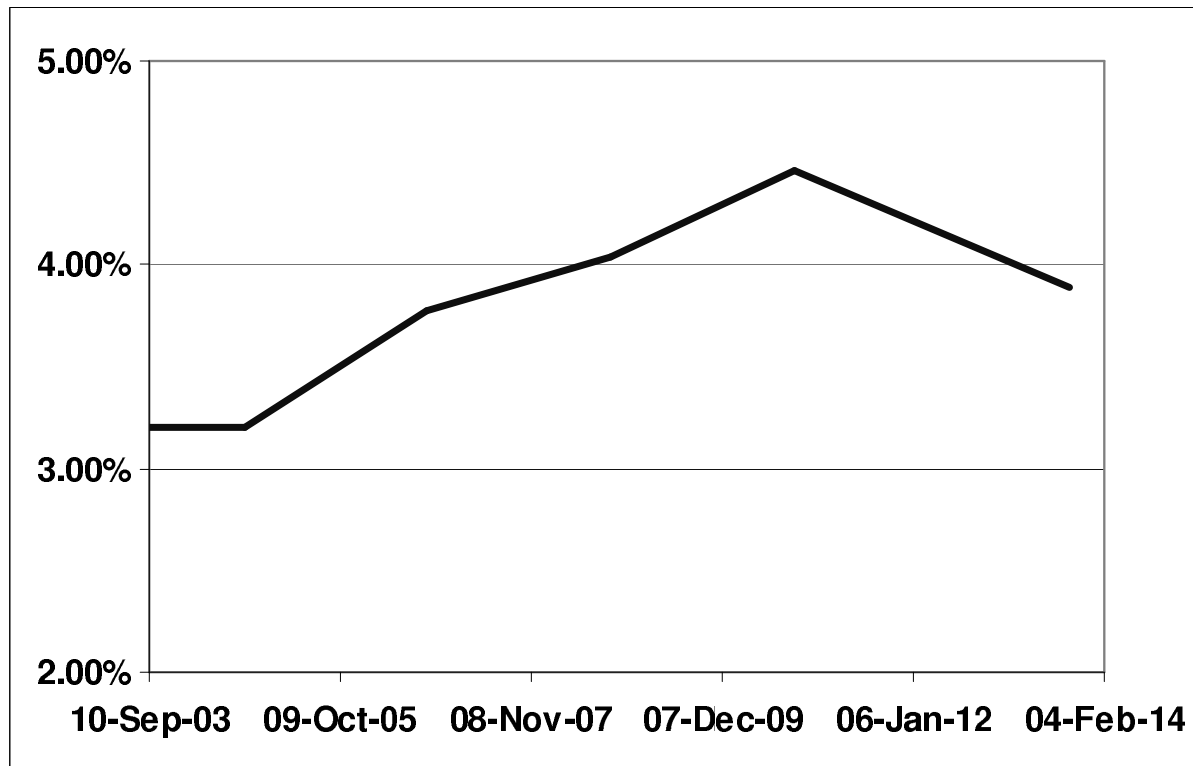


Figure 1: Piecewise linear intensity $t \mapsto \gamma(t)$ calibrated on Parmalat CDS quotes on September 10th, 2003.

CDS's with deterministic intensities: A first Example

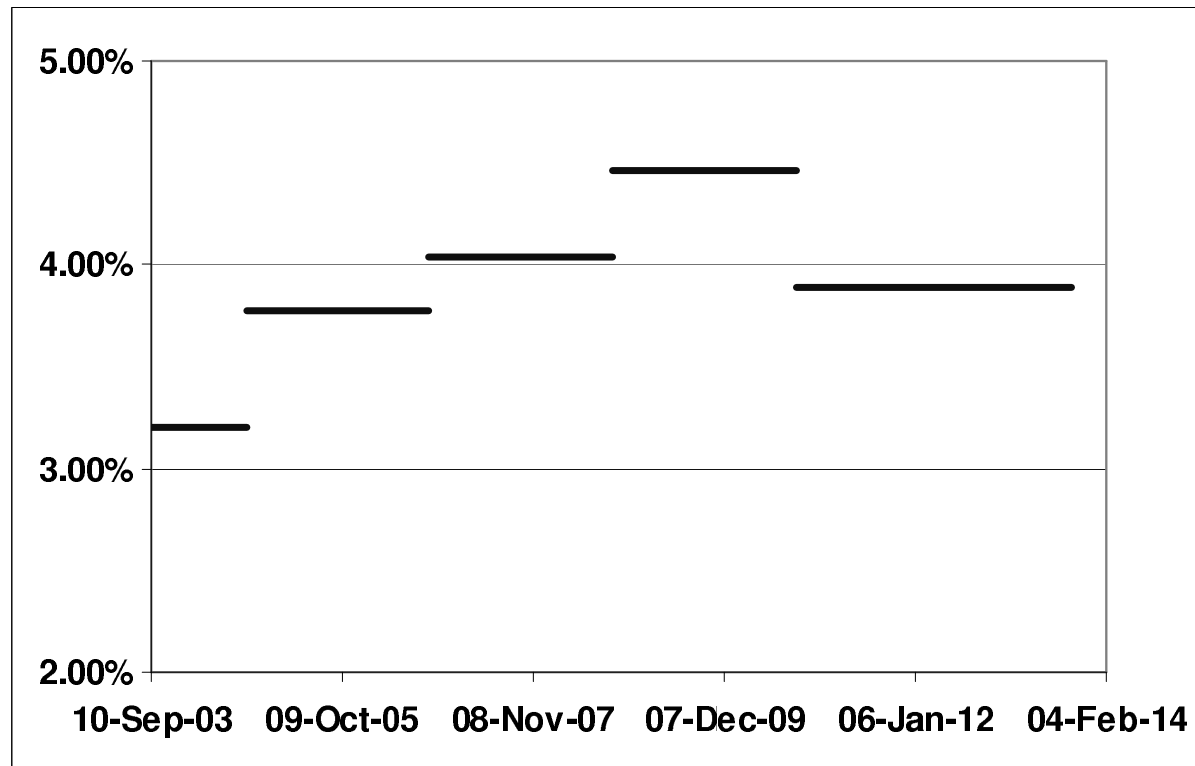


Figure 2: Piecewise constant intensity $t \mapsto \gamma(t)$ calibrated on Parmalat CDS quotes on September 10th, 2003.

CDS's with deterministic intensities: A first Example

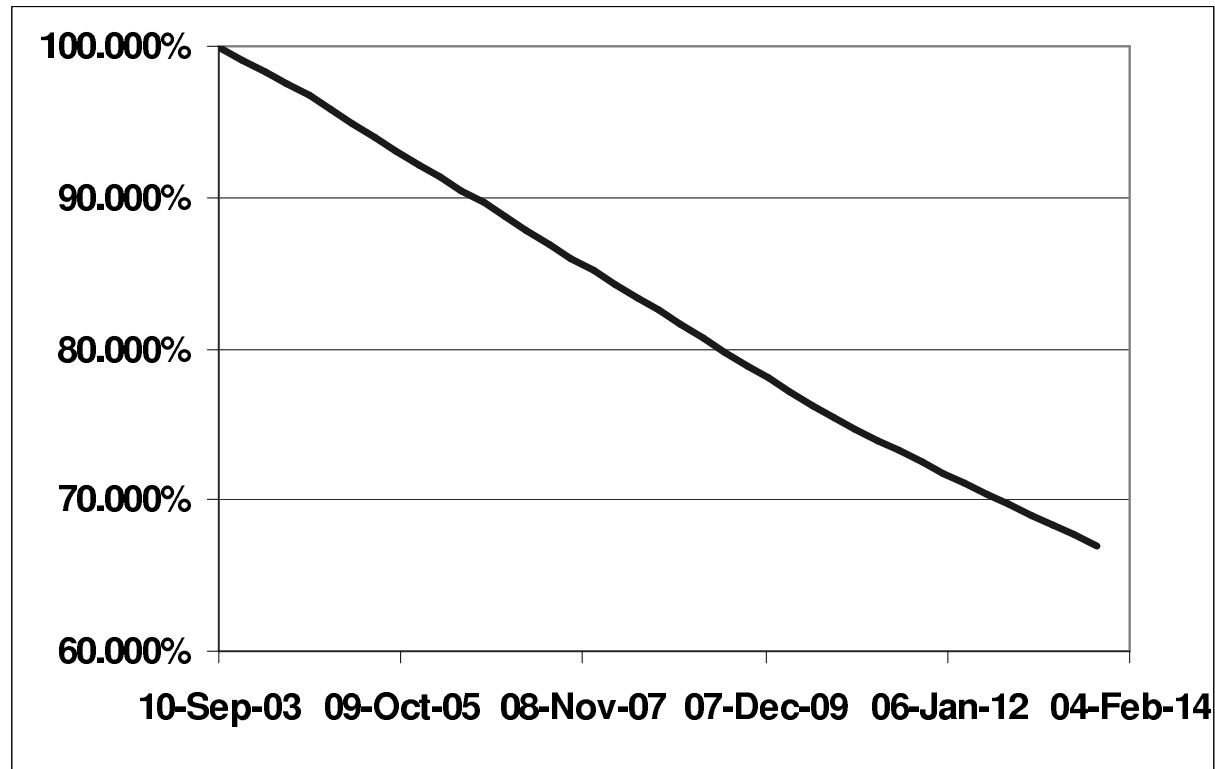


Figure 3: Survival probability $t \mapsto \exp(-\Gamma(t))$ resulting from calibration on CDS quotes on September 10th, 2003.

CDS's with deterministic intensities: Example

$\Gamma(t)$ and related prob ($\mathbb{Q}\{\tau < t\} = 1 - \exp(-\Gamma(t)) \approx \Gamma(t)$ for small Γ) obtained by calibrating the 1y, 3y, 5y, 7y and 10y CDS's on Merrill-Lynch. γ is piecewise linear.

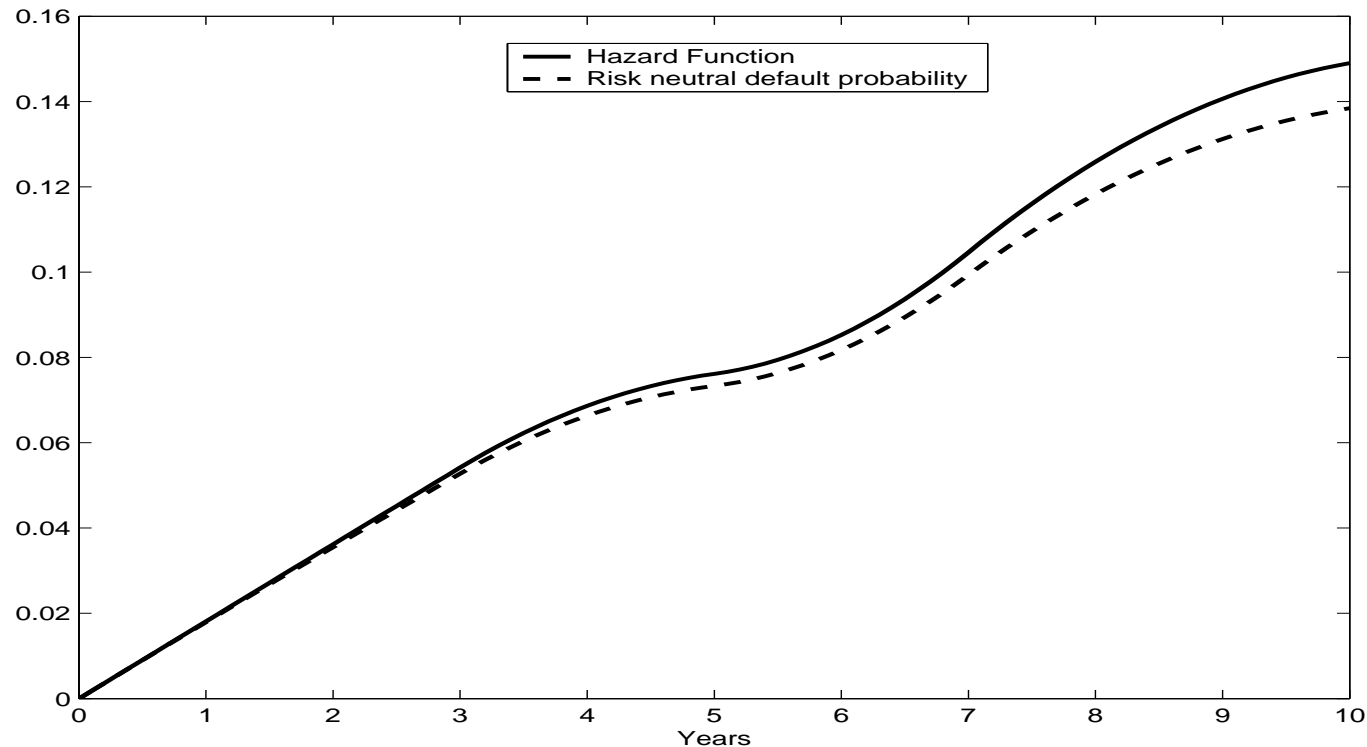


Figure 4: Hazard function and risk-neutral default probability for Merrill-Lynch CDS's, October 25, 2002

Defaultable Floaters and CDS's

Recall that without default the fair spread X making the Floating rate note quote at par is 0. Indeed, looking for the value of the spread X such that

$$0 = \mathbb{E} \left[-D(0, T_a) + \sum_{i=a+1}^b \alpha_i D(0, T_i) (L(T_{i-1}, T_i) + X) + D(0, T_b) 1 \right]$$

notional investment at T_a + periodic LIBOR + spread payments + final notional

we find $X = 0$. With Default the situation changes and the spread X that makes the defaultable floating rate note quote at par is strictly related to the CDS rate.

Defaultable Floating rate notes (DFRN) and CDS's

Indeed, recall the discounted payoff as $\Pi_{\text{DFRN}1_{a,b}} =$

$$-D(t, T_a)\mathbf{1}_{\{\tau > T_a\}} + \sum_{i=a+1}^b \alpha_i D(t, T_i) (L(T_{i-1}, T_i)\mathbf{1}_{\{\tau > T_i\}} + X \boxed{\mathbf{1}_{\{\tau > T_{i-1}\}}})$$

notional investment at T_a + periodic (LIBOR + spread) payments if no default yet

$$+ D(t, T_b)\mathbf{1}_{\{\tau > T_b\}} + \text{REC} \sum_{i=a+1}^b D(t, T_i)\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}}$$

+notional if no default +recovery in case of early default

The real DFRN payoff would have T_i in the box, but only with this change we find

$$\text{DFRN}1_{a,b}(t, X, \text{REC}) = \text{PRCDS}_{a,b}(t, X, 1 - \text{REC})$$

Defaultable Floaters and CDS's

$$\text{DFRN1}_{a,b}(t, X, \text{REC}) = \text{PRCDS}_{a,b}(t, X, 1 - \text{REC})$$

Looking for the fair CDS spread is the same as looking for the fair DFRN spread:

$$0 = \text{DFRN1}_{a,b}(t, X, \text{REC}) = \text{PRCDS}_{a,b}(t, X, 1 - \text{REC}) = 0$$

or in formula

$$X_{a,b}^{\text{FairDFRN}}(t) = R_{a,b}^{\text{PrCDS}}(t)$$

CDS Options and Callable Defaultable Floaters

At time T_a the holder of a (payer) CDS option has the right (but no obligation) to enter at maturity T_a a payer CDS with premium strike rate K for protection up to a later maturity T_b .

The value of this CDS at T_a is given by the protection leg minus the premium leg.

$$\text{PRCDS}_{a,b}(T_a, \text{LGD}, K) = \text{ ProtecLeg}_{a,b}(T_a, \text{LGD}) - \text{ PremiumLeg}_{a,b}(T_a, K)$$

The payoff of our option is given by said CDS if this is positive (we exercise the option), and zero otherwise (we do not exercise).

$$\Pi_{\text{CallCDS}_{a,b}}(T_a; K) = \left[\text{ ProtecLeg}_{a,b}(T_a, \text{LGD}) - \text{ PremiumLeg}_{a,b}(T_a, K) \right]^+$$

CDS Options and Callable Defaultable Floaters

$$\Pi_{\text{CallCDS}_{a,b}}(T_a; K) = [\text{ ProtecLeg}_{a,b}(T_a, \text{LGD}) - \text{ PremiumLeg}_{a,b}(T_a, K)]^+$$

By definition the protection leg is balanced, at time T_a , by a premium leg with fair premium rate $R_{a,b}(T_a)$, so we may write

$$\text{ ProtecLeg}_{a,b}(T_a, \text{LGD}) = \text{ PremiumLeg}_{a,b}(T_a, R_{a,b}(T_a)),$$

and our option payoff can be written as

$$\Pi_{\text{CallCDS}_{a,b}}(T_a; K) = [\text{ PremiumLeg}_{a,b}(T_a, R_{a,b}(T_a)) - \text{ PremiumLeg}_{a,b}(T_a, K)]^+$$

CDS Options and Callable Defaultable Floaters

$$\Pi_{\text{CallCDS}_{a,b}}(T_a; K) = \left[\text{PremiumLeg}_{a,b}(T_a, R_{a,b}(T_a)) - \text{PremiumLeg}_{a,b}(T_a, K) \right]^+$$

The premium leg price at time T_a through model-independent risk neutral valuation (more on this later), for a premium rate X known at T_a , is given by

$$\text{PremiumLeg}_{a,b}(T_a, X) = 1_{\{\tau > T_a\}} \sum_{i=a+1}^b \alpha_i \bar{P}(T_a, T_i) X$$

neglecting for the time being the accrual term. The option payoff then reads

$$\Pi_{\text{CallCDS}_{a,b}}(T_a; K) = 1_{\{\tau > T_a\}} \left[\sum_{i=a+1}^b \alpha_i \bar{P}(T_a, T_i) R_{a,b}(T_a) - \sum_{i=a+1}^b \alpha_i \bar{P}(T_a, T_i) K \right]^+$$

CDS Options and Callable Defaultable Floaters

$$\Pi_{\text{CallCDS}_{a,b}}(T_a; K) = 1_{\{\tau > T_a\}} D(t, T_a) \left[\sum_{i=a+1}^b \alpha_i \bar{P}(T_a, T_i) \right] (R_{a,b}(T_a) - K)^+$$

This is also the **exact** payoff if we use the first type postponed CDS, where R is replaced by R^{PR} (postponed running). The term between square brackets is often called **Annuity** or **PVBP** (Present Value per Basis Point), or sometimes **Risky Duration**. We thus have

Survival at maturity * Discount * [Annuity] * (Call option payoff on CDS Rate)

Notice the analogies with the swaption payoff in the interest rate market:

Discount * [Annuity or PVBP] * (Call option payoff on Swap Rate)

CDS Options and Callable Defaultable Floaters

Examples of CDS implied volatilities (The Model and exact definition of CDS volatility will be given later on). 3 companies C1 (Telephone, Baa2/BBB+) , C2 (Automotive, A3/BBB) and C3 (Telephone Baa2/BBB+) and CDS options quotes on March 26, 2004; $REC = 0.4$; $LGD = 1 - 0.4 = 0.6$; $T_0 = \text{March 26 2004 (0)}$; $T_a = \text{June 20 2004 (86d} \approx 3\text{m)}$ and $T'_a = \text{Dec 20 2004 (269d} \approx 9\text{m)}$; $T_b = \text{june 20 2009 (5y87d)}$; we consider receiver option quotes (puts on R) in basis points .

	Option: bid	mid	ask	$R_{0,b}(0)$	$R_{a,b}^{PR}(0)$	$R_{a,b}^{PR2}(0)$	K	$\sigma_{a,b}^{PR}$	$\sigma_{a,b}^{PR(2)}$
C1(T_a)	14	24	34	60	61.497	61.495	60	50.31	50.18
C2	32	39	46	94.5	97.326	97.319	94	54.68	54.48
C3	18	25	32	61	62.697	62.694	61	52.01	51.88
C1(T'_a)	28	35	42	60	65.352	65.344	61	51.45	51.32

Implied volatilities are rather high when compared with typical swaption volatilities. Same order of magnitude found by Hull and White (2003) via historical estimation. The option prices differ considerably, but the implied volatilities are similar. Changing postponement from PR to PR2 (maintaining the same $R_{0,b}(0)$'s and re-stripping Γ 's) leaves both CDS forward rates and implied volatilities almost unchanged

Constant Maturity CDS

Constant Maturity CDS: Definition. Consider a contract protecting in $[T_a, T_b]$ against default of a reference credit “C”.

If default occurs in $[T_a, T_b]$, a protection payment LGD is made from the protection seller “B” to the protection buyer “A” at the first T_j following the default time.

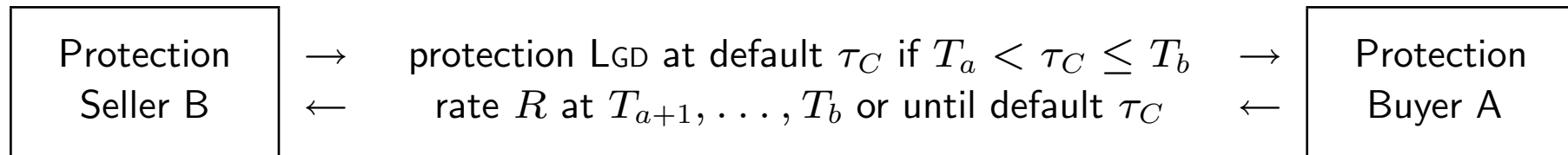
This is called “protection leg”.

In exchange for this protection “A” pays to “B” at each T_j before default a “ $c+1$ -long” (constant maturity) CDS rate $R_{j-1, j+c}(T_{j-1})$, with “ c ” integer > 0 .

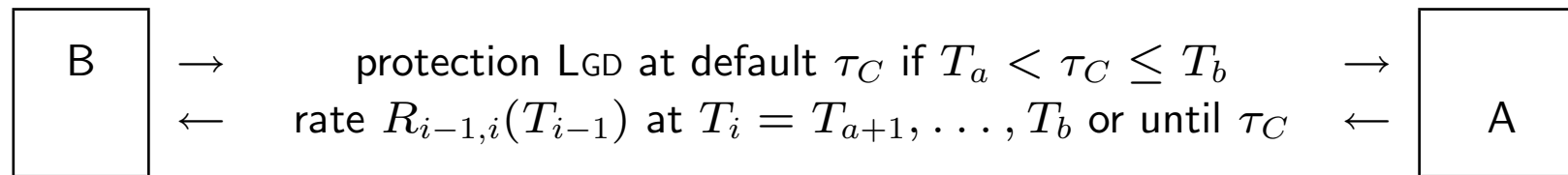
Notice that a product with $c = 0$ would be fair (initial price = 0) and equivalent to a standard CDS (more on this later)

Constant Maturity CDS

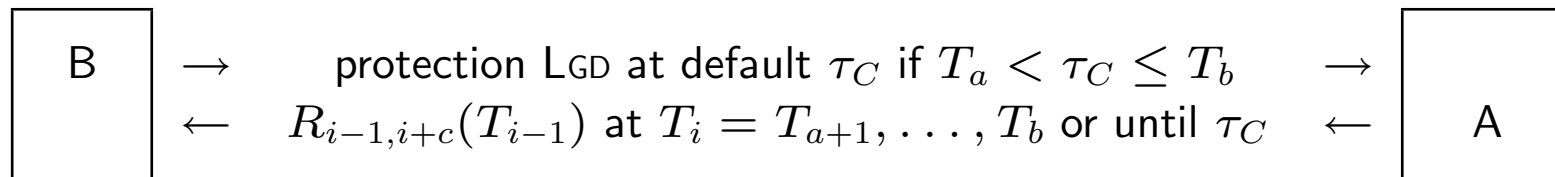
Standard CDS:



“Floating Rate Equivalent” of Standard CDS:



CMCDS:



Equity/Interest Rate Payoffs with Counterparty Risk

To simplify, we assume we are default-free and our counterparty is defaultable.

When we include counterparty risk in the valuation of an otherwise default-free derivative, in a way we obtain a credit derivative.

This holds for example for interest-rate swaps, for equity options and for equity return swaps, among other products.

In general, we can say that, if $\Pi(s, t)$ is a discounted payoff at time s , condensing all future discounted random payoffs from s to t , discounted back at s , with latest maturity T_b , we have that, including counterparty risk, we obtain as new discounted payoff:

$$\Pi^D(0, T_b) = \mathbf{1}_{\{\tau > T_b\}} \Pi(0, T_b) + \mathbf{1}_{\{\tau < T_b\}} \left[\Pi(0, \tau) + \text{REC}D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T_b))^+ - D(0, \tau) (-\mathbb{E}_\tau \Pi(\tau, T_b))^+ \right]$$

Equity/Interest-Rate Payoffs with Counterparty Risk

$$\Pi^D(0, T_b) = \mathbf{1}_{\{\tau > T_b\}} \Pi(0, T_b) + \mathbf{1}_{\{\tau < T_b\}} \left[\Pi(0, \tau) + \text{REC} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T_b))^+ - D(0, \tau) (-\mathbb{E}_\tau \Pi(\tau, T_b))^+ \right]$$

Original payoff if default after final maturity + (in case of early default) due payments before default + recovery of the residual NPV at default if this is positive - total residual NPV at default if this is negative.

It is possible to prove that the value of this discounted payoff is, in general, the same as the value of the following payoff:

$$\Pi_D(0, T_b) := \Pi(0, T_b) - \text{LGD} \mathbf{1}_{\{\tau < T_b\}} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T_b))^+$$

Original payoff - LGD call option with zero strike on residual NPV at default

The inclusion of counterparty risk adds a level of optionality to the payoff.

Equity/Interest-Rate Payoffs with Counterparty Risk

Example: Payer Interest Rate Swap.

$$\Pi(t, T_b) = \sum_{i=\beta(t)+1}^b D(t, T_i) \alpha_i (L(T_{i-1}, T_i) - K) \quad (3)$$

and the forward swap rate rendering the contract fair at time t is

$$K = S(t; T_{\beta(t)}, T_b) = \frac{P(t, T_{\beta(t)}) - P(t, T_b)}{\sum_{i=\beta(t)+1}^b \alpha_i P(t, T_i)}. \quad (4)$$

With this K , the value of the IRS contract at time t is 0.

Equity/Interest-Rate Payoffs with Counterparty Risk

With general strike K , the payoff can be rewritten as

$$\Pi(t, T_b) = D(t, T_{\beta(t)}) \sum_{i=\beta(t)+1}^b \alpha_i P(T_{\beta(t)}, T_i) (S(T_{\beta(t)}, T_{\beta(t)}, T_b) - K) \quad (5)$$

What is the value of an IRS under counterparty risk? We can write

$$\begin{aligned} \Pi_D(0, T_b) &= \Pi(0, T_b) - \text{LGD} \mathbf{1}_{\{\tau < T_b\}} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T_b))^+ = \\ &= \Pi(0, T_b) - \text{LGD} \sum_{i=1}^b \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T_b))^+ \end{aligned}$$

Equity/Interest-Rate Payoffs with Counterparty Risk

Let us postpone the default event to the first T_i following τ :

$$\begin{aligned}
 &\approx \Pi(0, T_b) - \text{LGD} \sum_{i=1}^b \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}} D(0, T_i) (\mathbb{E}_{T_i} \Pi(T_i, T_b))^+ \\
 &= \Pi(0, T_b) - \text{LGD} \sum_{i=1}^b \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}} D(0, T_i) \sum_{j=i+1}^b \alpha_j P(T_i, T_j) (S(T_i; T_i, T_b) - K)^+ \\
 &= \text{IRS}(0, T_b, K) - \text{LGD} \sum_{i=1}^b \mathbf{1}_{\{\tau \in (T_{i-1}, T_i]\}} D(0, T_i) \text{PayerSwaptionPayoff}(T_i, T_b, K) \\
 &\text{DefaultFreeIRS} - \text{LGD} * \text{stream of default probability weighted Swaptions}
 \end{aligned}$$

Equity/Interest-Rate Payoffs with Counterparty Risk

Example: **Equity call option**. S^C : Equity of firm “C”. Buy Call Option on S^C from “B”. Let $\tau = \tau_B$ be the default time of the counterparty. Payoff without counterparty risk:

$$\Pi(0, T) = D(0, T)(S_T^C - K)^+$$

With counterparty risk:

$$\Pi_D(0, T) = D(0, T)(S_T^C - K)^+ - \text{LGD} \mathbf{1}_{\{\tau_B < T\}} D(0, \tau_B) (\mathbb{E}_{\tau_B} [D(\tau_B, T)(S_T^C - K)^+])$$

Through iterated expectations, this payoff has the same value at time 0 as the payoff

$$\Pi_{D2}(0, T) = D(0, T)[1 - \text{LGD} \mathbf{1}_{\{\tau_B < T\}}](S_T^C - K)^+ = D(0, T)[\text{REC} \mathbf{1}_{\{\tau_B < T\}} + \mathbf{1}_{\{\tau_B > T\}}](S_T^C - K)^+$$

Equity/Interest-Rate Payoffs with Counterparty Risk

Example: **Equity return swap (aka Total rate of return swap).**

Initial Time 0: no flows, or

A \longrightarrow KS_0^C cash \longrightarrow B

A \longleftarrow K equity of "C" \longleftarrow B

....

Time T_i :

A \longrightarrow equity dividends of "C" \longrightarrow B

A \longleftarrow Libor + Spread \longleftarrow B

....

Final Time T_b :

A \longrightarrow K equity of "C" \longrightarrow B

A \longleftarrow KS_0^C cash \longleftarrow B

Equity/Interest-Rate Payoffs with Counterparty Risk

Example: **Equity return swap (aka Total rate of return swap)**.

Assume we are a company “A” entering a contract with company “B”, our counterparty. The reference underlying equity is company “C”.

Companies “A” and “B” agree on a certain amount K of stocks of a reference entity “C” (with price S) to be taken as nominal ($N = K S_0$). The contract starts in $T_a = 0$ and has final maturity $T_b = T$.

At $t = 0$ there is no exchange of cash (alternatively, we can think that “B” delivers to “A” an amount K of “C” stock and receives a cash amount equal to $K S_0$).

At intermediate times “A” pays to “B” the dividend flows of the stocks (if any) in exchange for a periodic rate (for example a semi-annual LIBOR or EURIBOR rate L) plus a spread X .

At final maturity $T = T_b$, “A” pays $K S_T$ to “B” (or gives back the amount K of stocks) and receives a payment $K S_0$.

Equity/Interest-Rate Payoffs with Counterparty Risk

The price of this product can be derived using risk neutral valuation, and the (fair) spread is chosen in order to obtain a contract whose value at inception is zero. We assume deterministic interest rates and deterministic known dividends. The discounted payoff to “A”, without counterparty risk, would be

$$\Pi(0, T) = -K \text{NPV}_{dividends}^{0 \div T_b}(0) + K S_0 \sum_{i=1}^b D(0, T_i) \alpha_i (L(T_{i-1}, T_i) + X) + D(0, T_b) (K S_0 - K S_{T_b})$$

It can be proved that if we do not consider default risk for “B” and “C”, the fair spread is identically equal to zero, in that

$$\mathbb{E}_0 \Pi(0, T) = X K S_0 \sum_{i=1}^b \alpha_i P(0, T_i) = 0 \text{ for } X = 0$$

Equity/Interest-Rate Payoffs with Counterparty Risk

However, when taking into account counterparty default risk of “B” in the valuation the fair spread is no longer zero. Using our usual formula and calling $\tau = \tau_B$, $S = S^C$

$$\Pi_D(0, T_b) = \Pi(0, T_b) - \text{LGD} \mathbf{1}_{\{\tau < T_b\}} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T_b))^+$$

we obtain

$$\Pi_D(0, T_b) := X K S_0 \sum_{i=1}^b \alpha_i D(0, T_i) - \text{LGD} \mathbf{1}_{\{\tau < T_b\}} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T_b))^+$$

where

$$\begin{aligned} \Pi(\tau, T_b) = & -K \text{NPV}_{\text{dividends}}^{\tau \div T_b}(\tau) + K S_0 \sum_{i=\beta(\tau)}^b D(\tau, T_i) \alpha_i (L(T_{i-1}, T_i) + X) \\ & + (K S_0 - K S_{T_b}) D(\tau, T_b) \end{aligned}$$

Equity/Interest-Rate Payoffs with Counterparty Risk

We ignore default of the underlying “C”, thus assuming it has a much stronger credit quality than the counterparty “B”.

This can be the case for example when “C” is an equity index.

In case an early default of the counterparty “B” occurs, the following happens. Let us call $\tau = \tau_B$ the default instant.

Before τ everything is as before, but if $\tau \leq T$, the net present value (NPV) of the position at time τ is computed. If this NPV is negative for us, i.e. for “A”, then its opposite is completely paid to “B” by us at time τ itself.

On the contrary, if it is positive for “A”, it is not received completely but only a recovery REC of that NPV is received by us.

It is clear that to us (“A”) the counterparty risk is a problem when the NPV is large and positive, since in case “B” defaults we receive only a fraction of it.

Equity Default Swaps

Equity Default Swaps (EDS) are a recent class of instruments mixing, in some sense, Equity and Credit features.

An EDS is typically a 5 years maturity OUT OF THE MONEY AMERICAN DIGITAL PUT struck at 30% of the spot price. The premium is not paid upfront but by means of a semiannual coupon stream.

In more colloquial terms: as soon as the underlying equity drops to or below 30% of its value at inception, a protection payment of some notional is made to the protection buyer. In exchange for this, the protection buyer pays the protection seller a premium rate on the notional, semiannually, up to five years or to the equity event if before.

We notice a clear analogy with CDS.

Equity Default Swaps

Recall the CDS:

Protection	"B"	→	protection LGD at default τ_C if $T_a < \tau_C \leq T_b$	→	"A"	Prot.
Seller	"B"	←	rate R at T_{a+1}, \dots, T_b or until default τ_C	←	"A"	Buyer

The EDS would read, if we define τ_E as the first time the equity S^C of name "C" goes below 30% of the current equity S_0 :

Protection	"B"	→	protection LGD_E at τ_E if $0 < \tau_E \leq 5y$	→	"A"	Prot.
Seller	"B"	←	rate R_E at $6m, 1y, \dots, 5y$ or until τ_E	←	"A"	Buyer

Typically $\text{LGD}_E = 0.5$ (De Servigny, Gilkes and Jobst, 2005).

Equity Default Swaps

The main difference between EDS and CDS is in the variable triggering default. If we adopt a first passage model for credit derivatives for example, where default is triggered by the value of the firm V hitting a safety barrier (more on this later), it is the firm value V and not the firm equity S that triggers default. S in general is an *option* on V , but this point of view is not adopted when pricing equity derivatives: in that case, one models S directly rather than considering its dynamics as induced by the option formula in V .

In general V and its barrier should be calibrated to CDS data (more on this later), while S should be calibrated to equity data (implied volatility, smile, etc).

If this were possible, EDS would be a purely equity product: The payoff is a standard barrier option payoff, only with a far out of the money strike; as such, it can be priced in principle with any equity smile model (local volatility CEV, local volatility displaced diffusion, Mixture Dynamics, SABR, Heston, Uncertain Vol....).

Equity Default Swaps

But it turns out EDS are not a purely equity product. Why so, if the payoff is equity and if the price can be in principle computed by means of an equity smile model?

The problem is in the calibration data for the smile model. There are no implied volatility data from the pure equity option market at strikes as small as $30\%S_0$. The only market products that may contain data on that region of the underlying are CDS. The idea is:

- a) Calibrate some of the parameters of a first passage dV model to CDS
- b) Compute the equity as an option on V : $S_t = \text{Option}(t, V_t)$
- c) Deduce S dynamics: $dS_t = d\text{Option}(t, V_t)$
- d) If possible, calibrate dS via remaining free parameters in dV to equity smile.
- e) Price the EDS with the resulting dS model, calibrated both to equity smile data and CDS data. Step “d” can be skipped if too difficult, and we price the EDS with an equity model entirely calibrated to credit data.

Equity Default Swaps

Why have EDS been introduced?

Actually, it can be checked that the premium rates R_E from EDS are typically much higher than the corresponding R from CDS (Brockhaus, 2005). In fact, the triggering event for EDS is more likely than the default event that triggers protection in CDS, so that a higher R is needed to balance it.

Also, at some point CDS had become a possible investment instrument with larger returns than most equity derivatives. To contrast this, equity people decided to mimic the CDS mechanism in an equity contest.