EXERCISE SET 2: INTENSITY MODELS and COPULAS for MULTI-NAME PRODUCTS.

Excercise 0: Review the exercises below and the related solutions, and signal possible mistakes in the solutions proposed, motivating.

EXERCISE 1: CIR intensity.

EXERCISE 1. Assume we are given a stochastic intensity process of CIR type,

$$d\lambda_t = \kappa(\mu - \lambda_t)dt + \nu\sqrt{\lambda_t}dW(t)$$

where $\lambda_0, \kappa, \mu, \nu$ are positive constants. W is a brownian motion under the risk neutral measure.

a) Under which conditions is the intensity strictly positive?

b) Increasing κ increases or decreases randomness in the intensity? And ν?

c) The mean of the intensity at future times is affected by k? And by ν?

d) What happens to mean of the intensity when time grows to infinity?

e) Is it true that, because of mean reversion, the variance of the intensity goes to zero (no randomness left) when time grows to infinity?

f) Can you compute a rough approximation of the percentage volatility in the intensity?

g) Suppose that $\lambda_0 = 400bps = 0.04$, $\kappa = 0.3$, $\nu = 0.001$ and $\mu = 400bps$. Can you guess the behaviour of the future random trajectories of the stochastic intensity after time 0?

h) Can you guess the spread of a CDS with 10y maturity with the above stochastic intensity when the recovery is 0.35?

EXERCISE 1: Solutions.

a) The condition for positivity is $2\kappa\mu > \nu^2$.

b) We can refer to the formulas for the mean and variance of λ_T in a CIR model as seen from time 0, at a given T. The formula for the variance is known to be (see for Example Brigo and Mercurio (2006))

$$\mathsf{VAR}(\lambda_T) = \lambda_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2$$

whereas the mean is

$$E[\lambda_T] = \lambda_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

We can see that for k becoming large the variance becomes small, since the exponentials decrease in k and the division by k gives a small value for large k. In the limit

$$\lim_{\kappa \to +\infty} \mathsf{VAR}(\lambda_T) = 0$$

so that for very large κ there is no randomness left.

We can instead see that $\mathrm{VAR}(\lambda_T)$ is proportional to ν^2, so that if ν increases randomness increases, as is obvious from $\nu\sqrt{\lambda_t}$ being the instantaneous volatility in the process λ.

c) As the mean is

$$E[\lambda_T] = \lambda_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

we clearly see that this is impacted by κ (indeed, "speed of mean reversion") and by μ clearly ("long term mean") but not by the instantaneous volatility parameter ν.

c) As T goes to infinity, we get for the mean

$$\lim_{T \to +\infty} \lambda_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T}) = \mu$$

so that the mean tends to μ (this is why μ is called "long term mean").

e) In the limit where time goes to infinity we get, for the variance

$$\lim_{T\to+\infty}[\lambda_0\frac{\nu^2}{\kappa}(e^{-\kappa T}-e^{-2\kappa T})+\mu\frac{\nu^2}{2\kappa}(1-e^{-\kappa T})^2]=\mu\frac{\nu^2}{2\kappa}$$

So this does not go to zero. Indeed, mean reversion here implies that as time goes to infinite the mean tends to μ and the variance to the constant value $\mu\frac{\nu^2}{2\kappa}$, but not to zero.

f) Rough approximations of the percentage volatilities in the intensity would be as follows. The instantaneous variance in $d\lambda_t$, conditional on the information up to t, is (remember that $VAR(dW(t)) = dt$)

$$VAR(d\lambda_t) = \nu^2\lambda_t dt$$

The percentage variance is

$$VAR\left(\frac{d\lambda_t}{\lambda_t}\right) = \frac{\nu^2\lambda_t}{\lambda_t^2}dt = \frac{\nu^2}{\lambda_t}dt$$

and is state dependent, as it depends on λ_t. We may replace λ_t with either its initial value λ_0 or with the long term mean μ, both known. The two rough percentage volatilities estimates will then be, for $dt = 1$,

$$\sqrt{\frac{\nu^2}{\lambda_0}} = \frac{\nu}{\sqrt{\lambda_0}}, \quad \sqrt{\frac{\nu^2}{\mu}} = \frac{\nu}{\sqrt{\mu}}$$

These however do not take into account the important impact of κ in the overall volatility of finite (as opposed to instantaneous) credit spreads and are therefore relatively useless.

g) First we check if the positivity condition is met.

$$2\kappa\mu = 2 \cdot 0.3 \cdot 0.04 = 0.024; \quad \nu^2 = 0.001^2 = 0.000001$$

hence $2\kappa\mu > \nu^2$ and trajectories are positive. Then we observe that the variance is very small: Take $T = 5y$,

$$\text{VAR}(\lambda_T) = \lambda_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \theta \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2 \approx 0.0000006.$$

Take the standard deviation, given by the square root of the variance:

$$\text{STDEV}(\lambda_T) \approx \sqrt{0.0000006} = 0.00077.$$

which is much smaller of the level 0.04 at which the intensity refers both in terms of initial value and long term mean. Therefore there is almost no randomness in the system as the variance is very small compared to the initial point and the long term mean.

Hence there is almost no randomness, and since the initial condition λ_0 is the same as the long term mean $\mu_0 = 0.04$, the intensity will behave as if it had the value 0.04 all the time. All future trajectories will be very close to the constant value 0.04.

h) In a constant intensity model the CDS spread can be approximated by

$$\lambda = \frac{R_{CDS}}{1 - REC} \Rightarrow R_{CDS} = \lambda(1 - REC) = 0.04(1 - 0.35) = 260bps$$

EXERCISE 2: Copula functions.

EXERCISE 2. Consider two standard normal random variables X_1, X_2 that are jointly normal with correlation ρ. Write the copula functions for the following values of ρ:

a) $\rho = 0$

b) $\rho = 1$

c) $\rho = 1/2$

d) Write the copula for the random vector (X_1, X_2^3) when $\rho = 1/2$.

EXERCISE 2: Solutions.

As the variables are standard normal, if we call Φ the cdf of the standard normal we know that

$$\Phi(X_1) = U_1, \Phi(X_2) = U_2$$

are uniform random variables. The copula is then defined as the multivariate distribution of the uniforms,

$$C(u_1, u_2) = \mathbb{Q}(U_1 \leq u_1, U_2 \leq u_2)$$

a) For $\rho = 0$ in two jointly standard gaussians, we know this corresponds to having independent random variables X_1 and X_2. It follows that $\Phi(X_1) = U_1, \Phi(X_2) = U_2$ are also independent, in that they are deterministic transforms of the independent X_1 and X_2.

We have therefore the copula as

$$C(u_1, u_2) = \mathbb{Q}(U_1 \leq u_1, U_2 \leq u_2) = \ by\ independence\ = \mathbb{Q}(U_1 \leq u_1)\mathbb{Q}(U_2 \leq u_1) = u_1 u_2$$

given that the uniform cdf is $\mathbb{Q}(U \leq u) = u$.

b) For $\rho = 1$ in two jointly standard gaussians, we know this corresponds to having total dependence random variables X_1 and X_2, i.e. $X_1 = X_2$. It follows that $\Phi(X_1) = U_1,\ \Phi(X_2) = \Phi(X_1) = U_1$.

We have therefore the copula as

$$C(u_1, u_2) = \mathbb{Q}(U_1 \leq u_1, U_1 \leq u_2) = \mathbb{Q}(U_1 \leq u_1 \ and \ U_1 \leq u_2)$$

$$= \mathbb{Q}(U_1 \leq \min(u_1, u_2)) = \min(u_1, u_2)$$

given that the uniform cdf is $\mathbb{Q}(U_1 \leq u) = u$.

c) For $\rho = 1/2$ we cannot invoke a special calculation; we obtain just the Gaussian copula for correlation parameter $1/2$, that cannot be written in closed form but only as an integral of the related bivariate density.

d) We know the copula is invariant for transformations that preserve information, i.e. invertible. Since

$$(X_1, X_2) \mapsto (X_1, X_2^3)$$

is invertible, with inverse

$$(Y_1, (Y_2)^{1/3}) \longleftarrow (Y_1, Y_2)$$

we have that the copula is the same as in point c) above.

EXERCISE 3: CDO tranche mechanics.

EXERCISE 3. An investor sells protection on the 12%-22% tranche of the i-Traxx pool. We assume recovery $\text{Rec} = 40\%$ for all names. Each single name in the portfolio has a credit position in the index of $1/125 = 0.8\%$ and participates to the aggregate loss in terms of 0.8% x Lgd $=$ 0.8% x $(1 - \text{Rec}) = 0.8\% \times 0.6 = 0.48\%$. This means that each default corresponds to a loss of 0.48% in the global portfolio.

a) How many defaults are needed because the tranche starts experiencing some losses?

b) How many defaults are needed for the tranche to be completely wiped out?

c) Assume we want to compute the periodic spread or quarterly insurance premium rate for the tranche $70\% - 90\%$. Can we say something about this tranche spread without precise information on correlation?

EXERCISE 3: Solutions.

a) For the loss to be equal to at least 12%, thus impacting the tranche, we need to have default of n names where

$$n \times \frac{1}{125} \times (1 - 0.4) = n \times 0.48\% > 12\%$$

This implies

$$n > 12/0.48 = 25.$$

Then the tranche will be impacted from the 26th default on.

b) For the tranche to be completely wiped out we need to have default of N names where

$$N \times \frac{1}{125} \times (1 - 0.4) = N \times 0.48\% > 22\%$$

This implies

$$N > 22/0.48 = 45.83$$

Therefore the $12 - 22\%$ tranche will be completely wiped out starting from the 46-th default.

c) The maximum loss that the pool can experience is when all 125 names default. In this case the total loss is

$$125 \times \frac{1}{125} \times (1 - 0.4) = 0.6 = 60\%$$

This means that with recovery 0.4 the loss will never reach 70%, not even if the whole portfolio defaults.

Since there can be no default in the default leg of the tranche, the value of the default leg of the tranche is 0.

Since the spread of the tranche is defined as the default leg divided by the tranche DV01, and the latter is nonzero, we have that the resulting spread is zero. The only way in the above setup to have the tranche being impacted at all, is to lower the recovery.

EXERCISE 4: First to default basket.

EXERCISE 4. We are given three CDS on names 1,2,3. Assume each single name default is modeled with a constant hazard rate and that the CDS premium legs pay continuously, so that the hazard rate formulas

$$\lambda_i = \frac{R_{CDS,i}}{1 - \text{Rec}_i}, \quad i = 1, 2, 3$$

hold for the three default probabilities.

Assume the three names have all recovery 20% and they have spreads respectively

$$R_{CDS,1} = 100bps, \ R_{CDS,2} = 200bps, \ R_{CDS,3} = 300bps$$

a) If the three defaults are connected through an independence (orthogonal) copula, and risk-free interest rates are zero, compute the default leg price of a first to default basket on the three names with a one year maturity.

b) Suppose the CDS spreads are now very small. Find a further approximation for the default leg price.

c) If the three defaults are connected through a maximum dependence copula C^+, compute the default leg price of a first to default basket on the three names with a one year maturity.

d) In which of the two cases a) and c) above is protection more expensive? Motivate

EXERCISE 4: Solution.

a) If i_1 is the random variable indexing the first default time $\tau^1 = \tau_{i_1}$, the default leg is

$$E[\text{LGD}_{i_1} D(0, \tau^1) 1_{\{\tau^1 \le 1y\}}]$$

Now, since all $\text{LGD} = 1 - \text{REC} = 1 - 0.2 = 0.8$ and $D(0,t) = 1$ for all t being interest rates zero, we have

$$0.8 E[1_{\{\tau^1 \le 1y\}}] = 0.8 \mathbb{Q}(\tau^1 \le 1) = 0.8(1 - \mathbb{Q}(\tau^1 > 1))$$

Now, $\mathbb{Q}(\tau^1 > 1)$ is the probability that the smallest of the default times is larger than 1. But saying that the smallest is larger than one is the same as saying that all are larger than one. So

$$\mathbb{Q}(\tau^1 > 1) = \mathbb{Q}(\tau_1 > 1 \ and \ \tau_2 > 1 \ \ and \ \ \tau_3 > 1) = \ldots$$

Since the defaults are independent, the probability becomes the product of probabilities,

$$\mathbb{Q}(\tau^1 > 1) = \mathbb{Q}(\tau_1 > 1)\mathbb{Q}(\tau_2 > 1)\mathbb{Q}(\tau_3 > 1) = e^{-\lambda_1 \cdot 1} e^{-\lambda_2 \cdot 1} e^{-\lambda_3 \cdot 1} = e^{-\lambda_1 - \lambda_2 - \lambda_3}$$

We then substitute and get the price of the default leg as

$$0.8(1 - \mathbb{Q}(\tau^1 > 1)) = 0.8(1 - e^{-\lambda_1 - \lambda_2 - \lambda_3})$$

b) If the CDS spreads are very small also the intensities will be very small. We have, using

$$e^x \approx 1 + x$$

for small x,

$$0.8(1 - e^{-\lambda_1 - \lambda_2 - \lambda_3}) \approx 0.8(\lambda_1 + \lambda_2 + \lambda_3) = R_{CDS,1} + R_{CDS,2} + R_{CDS,3}$$

i.e. we get approximately the sum of the three CDS spreads as a cost of the default leg.

c) In general in the intensity model with constant hazard rates we can write

$$\tau_1 = \Lambda_1^{-1}(\xi_1) = \frac{\xi_1}{\lambda_1}, \quad \tau_2 = \frac{\xi_2}{\lambda_2}, \quad \tau_3 = \frac{\xi_3}{\lambda_3}$$

With a maximum dependence copula we have that the exponential triggers (on which the copula is acting) are totally dependent, i.e. $\xi_1 = \xi_2 = \xi_3 = \xi$. This means that

$$\tau_1 = \frac{\xi}{\lambda_1}, \tau_2 = \frac{\xi}{\lambda_2}, \tau_3 = \frac{\xi}{\lambda_3}$$

all with the same ξ. Hence we can see that the smallest τ_i is the one with the largest λ_i, as we are dividing the same ξ for three possible λ's. In our case the largest λ is λ_3 coming from the largest spread $R_{CDS,3}$, so that the smallest τ is τ_3. This is true for every possible scenario of ξ. Hence we have in this case

$$\tau^1 = \tau_3$$

in all scenarios. The default leg is then

$$0.8E[1_{\{\tau^1 \le 1y\}}] = 0.8E[1_{\{\tau_3 \le 1y\}}] = 0.8\mathbb{Q}(\tau_3 \le 1) = 0.8(1 - e^{-\lambda_3 \cdot 1})$$

d)Protection is more expensive in case b) since

$$1 - \exp(-\lambda_1 - \lambda_2 - \lambda_3) > 1 - \exp(-\lambda_3)$$

This is obvious: If the three names are totally correlated, in a way it is enough to buy protection from the riskiest one of them and one is also protected from the other two. Not so if the correlation is smaller: in that case we have a larger premium to pay.