

## **EXERCISE SET 1: SINGLE NAME PRODUCTS AND MODELS.**

**Excercise 0:** Review the exercises below and the related solutions, and signal possible mistakes in the solutions proposed, motivating.

## EXERCISE 1: CDS AND HAZARD RATES.

**EXERCISE 1: CDS AND HAZARD RATES.** Assume a CDS quoted spread is 400 basis points and the recovery is estimated to be 20%. Under the assumptions that i) the premium leg of the CDS pays continuously and ii) the hazard rate is constant, find

- a) The hazard rate itself
- b) The probability of defaulting in 3 years and the probability of surviving 5 years
- c) What is the probability of surviving an infinite time? Obtain it as a limit calculation
- d) If the recovery increases, but the spread remains the same, does the probability of default increase or decrease? motivate
- e) Is it true that if the CDS spread doubles and the recovery remains the same, then the probability of default in ten years doubles?

## EXERCISE 1: SOLUTIONS

a) We use the simple formula

$$\gamma = \frac{R_{CDS}}{1 - REC}$$

that holds for continuous payments in the premium leg and for constant hazard rates,  $\gamma(t) = \gamma$  for all  $t$ . In our example

$$\gamma = \frac{400bps}{1 - 0.2} = 0.04/0.8 = 0.05 = 5\%$$

b) The probability of surviving time  $T$  in a hazard rate model with constant hazard rate  $\gamma$  is

$$\mathbb{Q}(\tau > T) = e^{-\gamma T} = e^{-0.05 \cdot 5} = e^{-0.25} = \dots$$

which is the probability of survival for five years.

The probability of defaulting by time  $T$  is

$$\mathbb{Q}(\tau \leq T) = 1 - \mathbb{Q}(\tau > T) = 1 - e^{-\gamma T} = 1 - e^{-0.05 \cdot 3} = \dots$$

which is the probability of defaulting in three years.

c) Compute the probability of surviving an infinite time as a limit

$$\lim_{T \rightarrow +\infty} \mathbb{Q}(\tau > T) = \lim_{T \rightarrow +\infty} e^{-\gamma T} = 0$$

for positive  $\gamma$  as is known from simple limit theory.

d) Given the formula

$$\gamma = \frac{R_{CDS}}{1 - REC}$$

and that the probability of default by  $T$  is

$$\mathbb{Q}(\tau \leq T) = 1 - e^{-\gamma T} = 1 - e^{-\frac{R}{1-REC}T}$$

we see that if the recovery increases,  $1 - REC$  decreases, so that  $\frac{R}{1-REC}$  increases and

$-\frac{R}{1-REC}$  decreases, and  $e^{-\frac{R}{1-REC}T}$  still decreases since the exponential is increasing, and finally  $-e^{-\frac{R}{1-REC}T}$  increases as this is the opposite of a decreasing function. Adding 1 does not change things, so the function is increasing. Default probabilities increase if we increase the recovery keeping the CDS rate fixed.

e) This is not true: in fact if now  $R_{new} = 2R_{old}$  then  $\gamma_{new} = 2\gamma_{old}$  follows easily (why?) but this does not imply

$$1 - \exp(-\gamma_{new}10) = 2(1 - \exp(-\gamma_{old}10)).$$

Rather, it implies

$$1 - \exp(-\gamma_{new}10) = 1 - \exp(-2\gamma_{old}10)$$

which is quite different. Only for small  $\gamma_{old}$ , by expanding the exponential, we get something close to twice the old default probability.

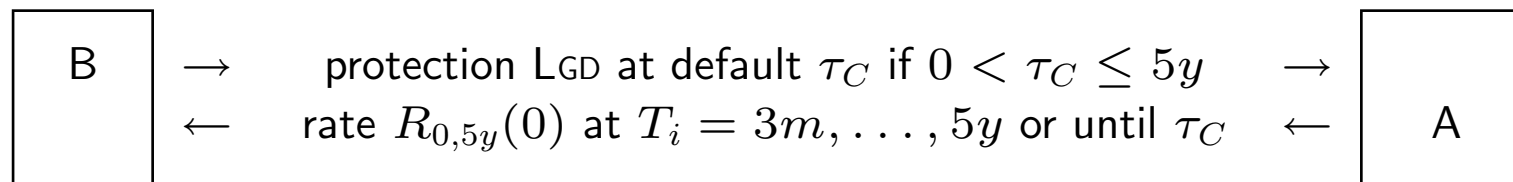
## EXERCISE 2: CDS and CMCDS.

2. Describe the product you obtain by buying protection up to five years with a standard CDS with fair running spread  $R_{0,5y}(0)$  fixed at time 0 and selling protection on the same name via a constant maturity CDS with a five year constant maturity in the running spread. Is this position subject to default risk or spread movement risk? Why?

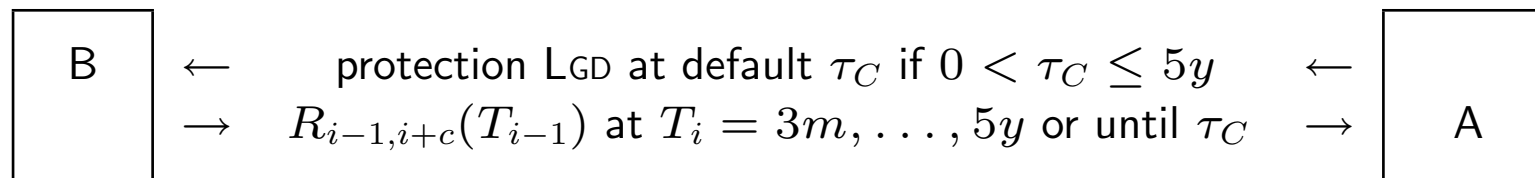
## EXERCISE 2: SOLUTION.

The position is described by the two diagrams, as seen from point of view of A who buys protection in the CDS (incoming protection arrow) and sells protection in the CMCDS (exiting protection arrow):

CDS standard

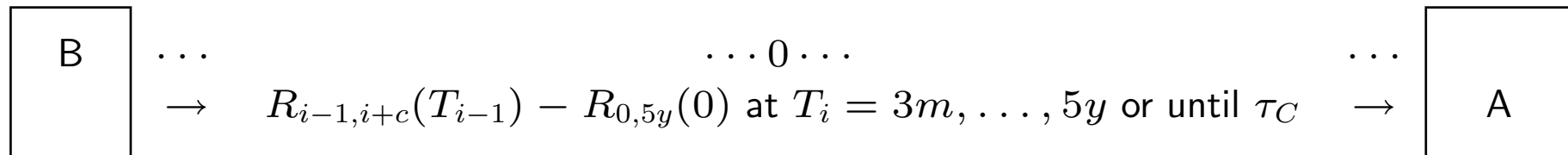


CMCDS:

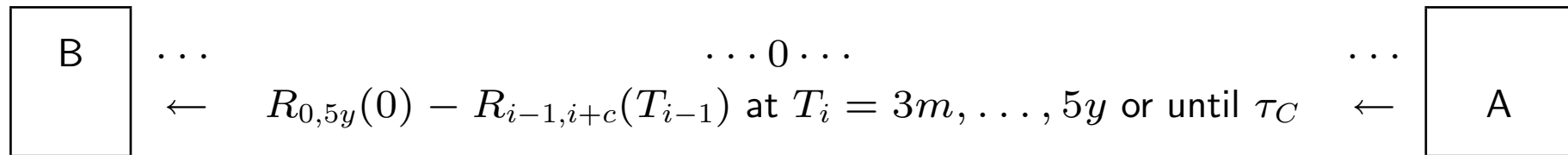


Since the two default or protection legs are the same, combining the two diagrams we obtain the following diagram of cash flows:

CDS - CMCDS:



or, equivalently, CDS - CMCDS:



the net effect is thus that we (we are "A") receive quarterly a spread that is  $R_{i-1,i+c}(T_{i-1}) - R_{0,5y}(0)$ , i.e. the difference between the five year fair CDS spread  $R_{i-1,i+c}(T_{i-1})$  prevailing at the coupon date minus the five year CDS spread  $R_{0,5y}(0)$

fixed initially. When we enter such a position it is then clear that if the CDS spread goes up in time, we will receive positive payments as  $R_{i-1,i+c}(T_{i-1}) - R_{0,5y}(0)$  will be positive,  $R_{i-1,i+c}(T_{i-1})$  having increased over the initial  $R_{0,5y}(0)$ . On the contrary, if CDS spreads go down, we will receive a negative difference, meaning that we actually have to pay the opposite (positive) amount to B. We are clearly subject to spread risk, but not to default risk as the default event itself does not impact us with the need to do a protection payment. This is because the default legs in the CDS and CMCDS are the same, so going long one and short the other one they cancel each other, which frees us from default payments risk.

## EXERCISE 3: STRUCTURAL MODELS.

**EXERCISE 3: STRUCTURAL MODELS.** In Merton's model, let the debt have final maturity  $T$  and the debt to be paid at maturity be  $L$ . Assume the firm value evolves like a geometric brownian motion

$$dV(t) = (r - k)V(t)dt + \sigma V(t)dW(t)$$

- a) Write a formula as seen from time 0 for the probability that the firm defaults (by  $T$ ).
- b) Is the Default probability increasing or decreasing in the debt level  $K$ ? Is the Default probability increasing or decreasing in the initial firm value level  $V(0)$ ? Motivate
- c) What happens if the initial firm value is very large (tends to infinity)? What is the behaviour of the default probability as a limit?

d) [difficult]. Assume for simplicity that  $r - k - \sigma^2/2 = 0$ . Suppose we want to compute the hazard rate in the merton model, as a limit for the probability of default by  $T$  divided by  $T$ , when  $T$  tends to zero. Show that this hazard rate tends to zero and compare with a standard intensity model with constant hazard rate. This says something important on "short term credit spreads" in the Merton model.

## EXERCISE 3: SOLUTIONS.

a) We know that the solution for the equation of  $V$  satisfies

$$\log V(T) = \log V(0) + (r - k - \sigma^2/2)T + \sigma W(T)$$

and

$$\log V(T) = \log V(0) + (r - k - \sigma^2/2)T + \sigma\sqrt{T}N(0, 1)$$

where  $N(0, 1)$  is a standard normal random variable. This implies that the default event, that in the Merton model can only happen at maturity, is written as  $\{V(T) \leq L\}$ , i.e. the final firm value went below the debt  $L$  to be paid. We have that  $\{V(T) \leq L\} = \{\log V(T) \leq \log L\}$  since  $\log$  is an increasing function. Given the above equation for  $V(T)$  this translates into

$$\{\log V(T) \leq \log L\} = \left\{ N(0, 1) \leq \frac{\log(L/V(0)) - (r - k - \sigma^2/2)T}{\sigma\sqrt{T}} \right\}$$

whose probability is

$$\mathbb{Q} \left\{ N(0, 1) \leq \frac{\log \frac{L}{V(0)} - (r - k - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right\} = \Phi \left( \frac{\log \frac{L}{V(0)} - (r - k - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right)$$

where  $\Phi$  is the cumulative distribution function of a  $N(0, 1)$  standard normal random variable.

b) Since  $\Phi$  is increasing, being a cdf, and  $\log$  is increasing, it follows that  $\log(L/V(0)) = \log L - \log V(0)$  is increasing in  $L$  and decreasing in  $V(0)$ , so that we can say the same for the default probability formula above.

c) If  $V(0) \rightarrow +\infty$  then  $\log(L/V(0)) \rightarrow -\infty$  and therefore since  $\lim_{x \rightarrow -\infty} \Phi(x) = 0$  we have that the default probability formula above tends to zero. This is intuitive: if  $V(0)$  is infinitely far away from the final debt level  $L$  then default will never happen, so that its probability will tend to zero.

d) To compute the hazard rate in Merton model we need to compute the limit of the above default probability formula, after substituting the simplification  $r - k - \sigma^2/2 = 0$ .

We have

$$\lim_{T \downarrow 0} \frac{\mathbb{Q}\{\tau \leq T\}}{T} = \lim_{T \downarrow 0} \frac{\Phi\left(\frac{\log \frac{L}{V(0)}}{\sigma\sqrt{T}}\right)}{T} =$$

This would lead to a limit of the type zero over zero. This is because in the numerator, the denominator tends to zero from the positive side, so that the fraction, with negative numerator, tends to minus infinity. Taking into account that the normal cdf  $\Phi$  tends to zero when the argument of the numerator tends towards minus infinity, we have a limit of the type zero over zero. We apply the limit theorem of De L'Hopital, and we get

$$= \lim_{T \downarrow 0} \frac{p\left(\frac{\log \frac{L}{V(0)}}{\sigma\sqrt{T}}\right) \left(-\frac{1}{2T\sqrt{T}} \frac{\log \frac{L}{V(0)}}{\sigma}\right)}{1} =$$

where  $p$  is the normal density function,

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

If we set  $1/\sqrt{T} = y$ , the limit becomes

$$= \lim_{y \uparrow +\infty} -y^3 \frac{\log \frac{L}{V(0)}}{2\sigma} p\left(y \frac{\log \frac{L}{V(0)}}{\sigma}\right) = A \lim_{y \uparrow +\infty} \frac{y^3}{\exp[(y^2(\log(L/V(0))/\sigma)^2)/2]} = 0$$

(for some constant A), since the exponential goes to infinity faster than any polynomial.

So we obtained that in the Merton model

$$\lim_{T \downarrow 0} \frac{\mathbb{Q}(\tau \leq T)}{T} = 0$$

Compare with a standard constant hazard rate model, where

$$\mathbb{Q}(\tau \leq T) = 1 - e^{-\gamma T}.$$

In this case we can see that the analogous limit gives

$$\lim_{T \downarrow 0} \frac{\mathbb{Q}\{\tau \leq T\}}{T} = \gamma$$

This is an important difference: basic structural models like Merton have no short term credit spreads (the limit is zero). Intensity models instead have non-zero short term credit spread. This is a modeling advantage of intensity models. It means that for very short maturities the Merton model will have great difficulties in fitting nonzero spreads, whereas the intensity model will have no problem.

## EXERCISE 4: HAZARD RATES CALCULATIONS.

**EXERCISE 4: HAZARD RATES CALCULATIONS.** Assume that, following calibration to CDS, we have a piecewise constant hazard rate  $\gamma(t)$  that takes the following values.

$$\gamma(t) = 0.02 \text{ for } 0 \leq t < 1y$$

$$\gamma(t) = 0.04 \text{ for } 1y \leq t < 2y$$

$$\gamma(t) = 0.02 \text{ for } 2y \leq t < 3y$$

Compute:

- a) The probability of defaulting in one year, two years and three years.
- b) The probability of surviving in two years.
- c) assuming that the recovery is 0.5, interest rates are zero and the CDS sells protection from today to three years, compute the price of the default or protection leg of the CDS.

## EXERCISE 4: Solution

a) Probability of default in  $T$  year is

$$\mathbb{Q}(\tau \leq T) = 1 - \exp\left(-\int_0^T \gamma(t) dt\right) = ?$$

Compute

$$\int_0^1 \gamma(t) dt = \int_0^1 0.02 dt = 0.02 \cdot 1 = 0.02$$

since  $\gamma(t)$  is constant and equal to 0.02 in the integration interval. Compute

$$\int_0^2 \gamma(t) dt = \int_0^1 \gamma(t) dt + \int_1^2 \gamma(t) dt = \int_0^1 0.02 dt + \int_1^2 0.04 dt = 0.02 \cdot 1 + 0.04 \cdot 1 = 0.06$$

Analogously,

$$\int_0^3 \gamma(t) dt = 0.02 \cdot 1 + 0.04 \cdot 1 + 0.02 \cdot 1 = 0.08$$

We can now compute the probabilities of defaulting in 1, 2 and 3 years respectively as

$$1 - \exp(-0.02), 1 - \exp(-0.06), 1 - \exp(-0.08)$$

b) The probability of surviving in two years is

$$\mathbb{Q}(\tau > T) = \exp\left(-\int_0^T \gamma(t) dt\right)$$

for  $T = 2y$  and hence  $\exp(-0.06)$ .

c) The default leg price is  $\text{LGD}E[D(0, \tau)1_{\{\tau \leq 3y\}}]$ . However, since interest rates are zero,  $r = 0$ , all discounts are equal to one,  $D(s, u) = \exp(-r(u - s)) = \exp(0) = 1$ . Hence we have

$$\begin{aligned} \text{LGD}E[D(0, \tau)1_{\{\tau \leq 3y\}}] &= \text{LGD}E[1 \cdot 1_{\{\tau \leq 3y\}}] = \text{LGD}E[1_{\{\tau \leq 3y\}}] = \text{LGD}\mathbb{Q}(\tau \leq 3y) = \\ &= \text{LGD}(1 - \exp(-0.08)) = (1 - \text{REC})(1 - \exp(-0.08)) = 0.5(1 - \exp(-0.08)) \end{aligned}$$