SECOND PARTIAL EXAM, end of 2005

Exercise 1. Consider the following tenor structure: $T_0 = 6m,\ T_1 = 1y,\ T_2 = 1y6m, .., \ T_6 = 3y6m, ...$ Consider the associated forward LIBOR rates $F_i(t) = F(t; T_{i-1}, T_i)$, $i = 1, \ldots, 6$, whose instantaneous volatility we denote by $\sigma_i(t)$. Consider the Caplet volatilities $v_{T_{i-1}}^{\text{Caplet}} =: v_{i-1}$ for the caplet resetting at T_{i-1} with maturity T_i.

$$v_0 = 0.2,\ v_1 = 0.22,\ v_2 = 0.2,\ v_3 = 0.18,\ v_4 = 0.17,\ v_5 = 0.16,$$

1.a) Find the LIBOR model volatilities $\sigma_i(t)$ consistent with these data in case we assume

$$\sigma_i(t) = \psi_{i-k} \quad \text{for} \quad t \in [T_{k-1}, T_k] :$$

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	ψ_1	Dead	Dead	...	Dead
$F_2(t)$	ψ_2	ψ_1	Dead	...	Dead
$\vdots$	...	...	...	...	...
$F_M(t)$	ψ_M	ψ_{M-1}	ψ_{M-2}	...	ψ_1

Hint: Notice that the tenor is semiannual and not annual;

1.b) Write the evolution of the term structure of caplet volatilities up to 2y. What is particular about this evolution?

Hint: Do you really need a computation for this?

1.c) Given the same caplet volatilities v_i, compute the LIBOR model volatilities Φ_i under the assumption $\sigma_i(t) = \Phi_i$.

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	Φ_1	Dead	Dead	...	Dead
$F_2(t)$	Φ_2	Φ_2	Dead	...	Dead
⋮	...	...	...	...	...
$F_M(t)$	Φ_M	Φ_M	Φ_M	...	Φ_M

1.d) Write the evolution of the term structure of caplet volatilities up to 2y. What is particular about this evolution?

1.e) Assuming instantaneous correlation $\rho_{4,5} = 0.8$, compute the terminal correlation corr$(F_4(T_1), F_5(T_1))$ between the $2y - 2.5y$ and the $2.5y - 3y$ forward rates in one year.

1.f) Same calculation with the LIBOR model volatilities Φ_i as from point c).

1.g) Given the linear exponential parameterization for instantaneous correlation,

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty) \exp[-\beta |i - j|], \quad \beta \geq 0.$$

$\beta = 0.2$, find ρ_∞ such that the above terminal correlation is 0.5 when volatilities are given by the ψ's in point a);

SECOND PARTIAL EXAM, end of 2005 (continued)

Exercise 2 Say whether the following statements are true or false and motivate your answer:

a) If forward LIBOR volatilities are constant in time, $\sigma_i(t) = \Phi_i$, $\sigma_j(t) = \Phi_j$, then the terminal correlation corr$(F_i(T_\alpha), F_j(T_\alpha))$ at a future time T_α is equal to the instantaneous correlation $\rho_{i,j}$;

b) If forward LIBOR volatilities are constant in time to expiry, like in the ψ parameterization, then the terminal correlation corr$(F_i(T_\alpha), F_j(T_\alpha))$ is equal to the instantaneous correlation $\rho_{i,j}$;

c) Terminal correlations may be larger, in absolute value, than the corresponding instantaneous correlations;

d) If forward LIBOR rates $F_{\alpha+1}(t), F_{\alpha+2}(t), \ldots, F_\beta(t)$ have a common value $\bar{F}$ then also the forward swap rate $S_{\alpha,\beta}(t)$ has the same value $\bar{F}$.

Exercise 3 Given the instantaneous squared volatily

$$\sigma_2(t)^2 = \exp(0.01t),$$

compute the $T_1 - T_2$ caplet volatility for $T_1 = 1y$ and $T_2 = 2y$;

Exercise 4 LIBOR CUBE. Consider a contract paying in $T_2 =$two years the cube of the LIBOR rate that reset in $T_1 =$one year. Compute the pricing formula of this product,

$$E[\frac{B(0)}{B(T_2)}(F_2(T_1))^3]$$

in a LIBOR market model setting

$$dF_2(t) = \sigma_2 F_2(t) dZ_2, \quad Q^2$$

as a function of the constant volatility σ_2, of the initial forward $F_2(0)$ and of the bond $P(0, T_2)$. This contract is not a classic option, but does this contract expose the owner to volatility risk?

(**Hint**: change numeraire and use the T_2 forward measure Q^2; Through Ito's formula, derive the dynamics for $(F_2)^3$ starting from the above dynamics for F_2; notice that $(F_2)^3$ follows again a geometric brownian motion under Q^2, so that you may compute this expectation with the usual formula for the expectation of a geometric brownian motion....)

Second partial exam: Solutions

EXERCISE 1

1.a) We know that the $T_0 - T_1$ caplet volatility in the LIBOR model is v_0, where

$$v_0^2 = \frac{1}{T_0}\int_0^{T_0} \sigma_1(t)^2 dt = \frac{1}{0.5}\int_0^{0.5} \psi_1^2 dt = \psi_1^2,$$

so that $\psi_1 = v_0$.

Similarly, the $T_1 - T_2$ caplet volatility is v_1, where

$$v_1^2 = \frac{1}{T_1}(\int_0^{T_1} \sigma_2(t)^2 dt) = \frac{1}{1y}(\int_0^{1y} \sigma_2(t)^2 dt) =$$

$$= (\int_0^{0.5y} \sigma_2(t)^2 dt + \int_{0.5y}^{1y} \sigma_2(t)^2 dt) = (\int_0^{0.5y} \psi_2^2 dt + \int_{0.5y}^{1y} \psi_1^2 dt)$$
$$= 0.5(\psi_1^2 + \psi_2^2),$$

from which

$$\psi_2 = \sqrt{2v_1^2 - \psi_1^2} = 0.1371$$

Then, analogously,

$$v_2^2 = \frac{1}{1.5}(0.5\psi_3^2 + 0.5\psi_2^2 + 0.5\psi_1^2)$$

and

$$\psi_3 = \sqrt{3v_2 - \psi_2^2 - \psi_1^2}$$

and so on. One obtains

ψ_1	0.2
ψ_2	0.238327506
ψ_3	0.152315462
ψ_4	0.09797959
ψ_5	0.122065556
ψ_6	0.09539392

1.b) Given instantaneous volatilities that are homogeneous in the time to expiry, such as ψ's, the evolution of the term structure is of stationary type, always equal to the initial caplet volatility structure, progressively losing the "tail" as time passes.

[Detailed solution omitted]

1.c) $\Phi_i = v_{i-1}$.

[Detailed solution omitted]

1.d) The term structure evolution is non-stationary, rolling down following its own "tail", starting from the initial caplet volatility term structure.

[Detailed solution omitted]

1.e)

$$\text{corr}(F_4(T_1), F_5(T_1)) = \rho_{4,5} \frac{\psi_4\psi_5 + \psi_3\psi_4}{\sqrt{(\psi_4^2 + \psi_5^2)(\psi_3^2 + \psi_4^2)}} =$$

$$= 0.8 * 0,948687 = 0.7586$$

1.f) For time-constant volatilities we know that the terminal correlation is equal to the instantaneous correlation, so

$$\text{corr}(F_4(T_1), F_5(T_1)) = \rho_{4,5} = 0.8$$

1.g) We impose

$$\text{corr}(F_4(T_1), F_5(T_1)) = 0.5$$

From the above calculations, this amounts to imposing

$$0.5 = \rho_{4,5} \frac{\psi_4\psi_5 + \psi_3\psi_4}{\sqrt{(\psi_4^2 + \psi_5^2)(\psi_3^2 + \psi_4^2)}}$$

or, given our parameterization for ρ,

$$0.5 = \rho_\infty + (1 - \rho_\infty) \exp[-0.2|4-5|] \frac{\psi_4\psi_5 + \psi_3\psi_4}{\sqrt{(\psi_4^2 + \psi_5^2)(\psi_3^2 + \psi_4^2)}}$$

Substituting the above ψ values and solving in ρ_∞ we obtain $\rho_\infty = -1.6077$. This is not an admissible correlation value, since it is smaller than minus one. This unrealistic and not admissible output is due to our unrealistic requirement on corr$(F_4(T_1), F_5(T_1))$ given the ψ values above. Two adjacent rates cannot have such a low correlation in a normal market situation.

EXERCISE 2

2.a) True, substitute the volatilities in the general terminal correlation formula to obtain

$$\text{corr}(F_i(T), F_j(T)) = \rho_{i,j} \frac{(\Phi_i \Phi_j)T}{\sqrt{(\Phi_i^2 T)(\Phi_j^2 T)}} = \rho_{i,j}$$

2.b) False, see exercise 1.e).

2.c) False, by Schwartz's inequality the opposite applies: Terminal correlations are always smaller or equal, in absolute value, than the corresponding instantaneous correlations;

2.d) Since swap rates are weighted averages of forward rates, if all forward rates are equal then the swap rate is equal to the common forward rate:

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) \bar{F} = 1\bar{F} = \bar{F}.$$

So the statement is true.

EXERCISE 3.

The squared caplet volatility is

$$v_1^2 = \frac{1}{T_1} \int_0^{T_1} \sigma_2(t)^2 dt$$

Since we have the integral $\int_a^b e^{-\alpha t} dt = \left. \frac{e^{-\alpha t}}{\alpha} \right|_a^b$ then

$$v_1 = \sqrt{\frac{1}{T_1} \left. \frac{e^{-0.01t}}{0.01} \right|_0^1} = \sqrt{\frac{e^{-0.01} - 1}{0.01}} \approx 1.0025$$

This is a huge volatility (100%), unrealistic in financial terms.

EXERCISE 4. (LIBOR CUBE).

We change the numeraire from B to the T_2 measure:

$$E^{\boxed{B}} \left[\frac{\boxed{B(0)}}{\boxed{B(T)}} F_2(T_1)^3 \right] = E^{\boxed{2}} \left[\frac{\boxed{P(0, T_2)}}{\boxed{P(T_2, T_2)}} F_2(T_1)^3 \right]$$

$$= P(0, T_2) E^2 [F_2(T_1)^3].$$

Since we know that the dynamics of F_2 under numeraire $P(.,T_2)$ is

$$dF_2 = \sigma_2 F_2 dZ,$$

by Ito's formula

$$d(F_2)^3 = 3(F_2)^2 dF_2 + \tfrac{1}{2} 3 \cdot 2 \cdot F_2 dF_2 dF_2$$

$$d(F_2)^3 = 3(F_2)^2 \sigma_2 F_2 dZ + 3F_2 \sigma_2^2 F_2^2 dZ dZ$$
$$d(F_2)^3 = 3(F_2)^3 \sigma_2 dZ + 3F_2^3 \sigma_2^2 dt$$

Set $Y_t = (F_2(t))^3$. From the last equation we have

$$dY = (3\sigma_2^2) Y dt + (3\sigma_2) Y dZ.$$

This is a "Black Scholes" geometric brownian motion of type

$$dY = aY dt + bY dZ$$

($a = 3\sigma_2^2,\ b = (3\sigma_2)$) whose expected value is

$$E[Y(T)] = Y_0 e^{aT}$$

Therefore the price is

$$P(0,T_2)E^2[F_2(T_1)^3] = P(0,T_2)E[Y(T_1)] = P(0,T_2)Y_0 e^{aT_1}$$

$$= P(0,T_2)F_2(0)^3 e^{3\sigma_2^2 T_1}$$

This last formula depends on the volatility σ_2, so this product does expose an investor to volatility risk.