EXERCISES (UPDATED!!), November 23, 2005 WITH SOLUTIONS

Exercise 0. Compare the results with your colleague students and find possible mistakes in the solutions attached below

Exercise 1. Consider the following tenor structure:

$$T_0 = 1y, T_1 = 2y, T_2 = 3y, \dots, T_5 = 6y.$$

Consider the associated forward LIBOR rates $F_i(t) = F(t; T_{i-1}, T_i)$, $i = 1, \dots, 6$, whose instantaneous volatility we denote by $\sigma_i(t)$. Consider the Caplet volatilities $v_{T_{i-1}}^{\text{Caplet}} =: v_{i-1}$ for the caplet resetting at T_{i-1} with maturity T_i.

a) Given the following caplet volatilities

$$\begin{aligned} v_0 &= 0.1; v_1 = 0.12; v_2 = 0.15; \\ v_3 &= 0.14; v_4 = 0.13; v_5 = 0.12 \end{aligned}$$

compute the LIBOR model vols $\sigma_i(t)$ consistent with these data in case we assume $\sigma_i(t) = \psi_{i-k}$ for $t \in [T_{k-1}, T_k]$:

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\dots$	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	ψ_1	Dead	Dead	$\dots$	Dead
$F_2(t)$	ψ_2	ψ_1	Dead	$\dots$	Dead
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$F_M(t)$	ψ_M	ψ_{M-1}	ψ_{M-2}	$\dots$	ψ_1

b) Repeat the same calculation if $v_5 = 0.08$ and notice an important point on the output you obtain.

c) Given the same input as in part a), compute the LIBOR model volatilities under the assumption $\sigma_i(t) = \Phi_i$.

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\dots$	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	Φ_1	Dead	Dead	$\dots$	Dead
$F_2(t)$	Φ_2	Φ_2	Dead	$\dots$	Dead
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$F_M(t)$	Φ_M	Φ_M	Φ_M	$\dots$	Φ_M

d) Again, repeat the same calculation as in c) but with $v_5 = 0.08$. Do you find the same problems as in point b) ?

e) Given the following separable volatilities, $\sigma_i(t) = \Phi_i \psi_{i-k}$ for $t \in [T_{k-1}, T_k)$,

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\dots$	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	$\Phi_1\psi_1$	Dead	Dead	$\dots$	Dead
$F_2(t)$	$\Phi_2\psi_2$	$\Phi_2\psi_1$	Dead	$\dots$	Dead
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$F_M(t)$	$\Phi_M\psi_M$	$\Phi_M\psi_{M-1}$	$\Phi_M\psi_{M-2}$	$\dots$	$\Phi_M\psi_1$

compute the caplet volatilities $v_1, \dots, v_5$:

$$\Phi_1 = 1, \quad \Phi_2 = 1.1, \quad \Phi_3 = 1.2, \quad \Phi_4 = 0.9, \quad \Phi_5 = 0.8, \quad \Phi_6 = 1.1$$

$$\psi_1 = 0.1, \quad \psi_2 = 0.12, \quad \psi_3 = 0.12, \quad \psi_4 = 0.09, \quad \psi_5 = 0.08, \quad \psi_6 = 0.11$$

f) With the same volatilities as in e), draw also the evolution of the term structure of caplet volatilities in time up to five years in the future.

g) Find the exponential Full rank instantaneous correlation structure

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)e^{-\beta|i-j|},$$

with $\beta = 0.2$ implying $\rho_{1,2} = 0.9$.

h) With instantaneous correlations as in g) and volatilities as in e), compute the terminal correlation matrix in two years (at time T_1) and say whether it looks acceptable.

Exercise 2

Consider the following curve of zero coupon bonds for the maturities

$T_0 = 1y, T_1 = 2y, \ldots, T_9 = 10y$:

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P(0,T0) 0.961538462 P(0,T1) 0.924556213 P(0,T2) 0.888996359
P(0,T3) 0.854804191 P(0,T4) 0.821927107
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a) Compute the forward swap rates $S_{1,4}(0)$ and $S_{2,4}(0)$.

b) Consider an option to enter a payer swap first resetting at $T_2 = 3y$ and lasting up to $T_4 = 5y$ with a fixed rate equal to $K = 5\%$. Let the volatility of the underlying swap rate be 20%. Compute the price of the related option.

c) Same as b) but the underlying swap is a receiver swap and the fixed rate is $K = 3\%$.

d) Compute the forward libor rates $F_1(0), F_2(0), \ldots$ and verify that the swap rate $S_{2,4}(0)$ is a weighted average of the forward LIBOR rates (in particular, compute the weights).

Exercise 3. Let S be the price of an asset. Let the dynamics of S follow the process

$$dS_t = rS_t dt + \sigma S_t dW_t^B,$$

under the risk neutral measure Q^B, where r and σ are positive constants and W^B is a one-dimensional brownian motion under Q^B (a particular "quadratic" option).

Consider a payoff ensuring the payment of the square of the price S at a final maturity T. Using the change of numeraire technique, derive the price of this contract at time 0. [Hint: take S itself as numeraire...]

Solutions will be posted by November 30

Solutions for the exercises of November 23, 2005

Exercise 1. a) We know that the $T_0 - T_1$ caplet volatility in the LIBOR model is v_0, where

$$v_0^2 = \frac{1}{T_0}\int_0^{T_0} \sigma_1(t)^2 dt = \frac{1}{1}\int_0^{1y} \psi_1^2 dt = \psi_1^2,$$

so that $\psi_1 = v_0 = 0.1$.

Similarly, the $T_1 - T_2$ caplet volatility is v_1, where

$$v_1^2 = \frac{1}{T_1}(\int_0^{T_1} \sigma_2(t)^2 dt) = \frac{1}{2y}(\int_0^{2y} \sigma_2(t)^2 dt) =$$

$$= \tfrac{1}{2}(\int_0^{1y} \sigma_2(t)^2 dt + \int_{1y}^{2y} \sigma_2(t)^2 dt) = \tfrac{1}{2}(\int_0^{1y} \psi_2^2 dt + \int_{1y}^{2y} \psi_1^2 dt)$$
$$= \tfrac{1}{2}(\psi_1^2 + \psi_2^2),$$

from which

$$\psi_2 = \sqrt{2v_1^2 - \psi_1^2} = 0.1371$$

Then, analogously,

$$v_2^2 = \frac{1}{3}(\psi_3^2 + \psi_2^2 + \psi_1^2)$$

and

$$\psi_3 = \sqrt{3v_2 - \psi_2^2 - \psi_1^2} = 0.1967,$$

and similarly

$$\psi_4 = 0.1044, \quad \psi_5 = 0.0781, \quad \psi_6 = 0.0436.$$

1. b). With $v_5 = 0.08$ we follow the same procedure, but when we reach

$$\psi_6^2 = \sqrt{6v_5^2 - \psi_5^2 - \psi_4^2 - \psi_3^2 - \psi_2^2 - \psi_1^2} = \sqrt{-0.0461} =$$

$$= 0.2147i$$

the square root of a negative number, i.e. an imaginary number, unacceptable as a volatility. Then in this case the chosen parameterization of volatilities cannot be calibrated to caplet volatilities.

1. c) As before, we know that the $T_0 - T_1$ caplet volatility in the LIBOR model is v_0, where

$$v_0^2 = \frac{1}{T_0}\int_0^{T_0} \sigma_1(t)^2 dt = \frac{1}{1}\int_0^{1y} \Phi_1^2 dt = \Phi_1^2,$$

so that $\Phi_1 = v_0 = 0.1$.

Similarly, the $T_1 - T_2$ caplet volatility is v_1, where

$$v_1^2 = \frac{1}{T_1}(\int_0^{T_1} \sigma_2(t)^2 dt) = \frac{1}{2y}(\int_0^{2y} \sigma_2(t)^2 dt) =$$

$$= \tfrac{1}{2}\left(\int_0^{1y} \sigma_2(t)^2 dt + \int_{1y}^{2y} \sigma_2(t)^2 dt\right) = \tfrac{1}{2}\left(\int_0^{1y} \Phi_2^2 dt + \int_{1y}^{2y} \Phi_2^2 dt\right)$$

$$= \tfrac{1}{2}(2\Phi_2^2) = \Phi_2^2,$$

from which

$$\Phi_2 = v_1 = 0.12$$

Then, analogously,

$$v_2^2 = \frac{1}{3}(\Phi_3^2 + \Phi_3^2 + \Phi_3^2)$$

and

$$\Phi_3 = v_2 = 0.15,$$

and similarly

$$\Phi_4 = v_3 = 0.14, \quad \Phi_5 = v_4 = 0.13, \quad \Phi_6 = v_5 = 0.12$$

d) This time $v_5 = 0.08$ gives no problem, since $\Phi_6 = v_5 = 0.08$ is fine. The Φ parameterization never breaks down if the input caplet volatilities make sense, whereas the ψ parameterization can give problems for steep caplet volatility graphs, as seen in case b) above.

e) By carrying out the related multiplications, $\sigma_i([T_{k-1}, T_k)) = \Phi_i \psi_{i-k}$, as in

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	$\Phi_1\psi_1$	Dead	Dead	...	Dead
$F_2(t)$	$\Phi_2\psi_2$	$\Phi_2\psi_1$	Dead	...	Dead
$\vdots$	...	...	...	...	...
$F_M(t)$	$\Phi_M\psi_M$	$\Phi_M\psi_{M-1}$	$\Phi_M\psi_{M-2}$	...	$\Phi_M\psi_1$

we build the σ table as follows:

0.1					
0.132	0.11				
0.144	0.144	0.12			
0.081	0.108	0.108	0.09		
0.064	0.072	0.096	0.096	0.08	
0.121	0.088	0.099	0.132	0.132	0.11

Then

$$v_1^2 = \frac{1}{T_1}(\int_0^{T_1} \sigma_2(t)^2 dt) = \frac{1}{2y}(\int_0^{2y} \sigma_2(t)^2 dt) =$$

$$= \tfrac{1}{2}(\int_0^{1y} \sigma_2(t)^2 dt + \int_{1y}^{2y} \sigma_2(t)^2 dt) =$$

$$= \tfrac{1}{2}(\int_0^{1y} (\Phi_2\psi_2)^2 dt + \int_{1y}^{2y} (\Phi_2\psi_1)^2 dt)$$

or, basically, the sum of the squares of the elements in the second row of the above matrix:

$$v_1^2 = \frac{1}{2}(0.132^2 + 0.11^2) = 0.01476, \quad v_1 = 0.1215$$

Similarly, v_2^2 is obtained by adding up the squares of the third row and dividing by 3:

$$v_2^2 = \frac{1}{3}(0.144^2 + 0.144^2 + 0.12^2) = 0.01862, \;\; v_2 = 0.07879$$

and so on.

f) Term structure of caplet volatilities in one year, i.e. at $T_0 = 1y$. From the formula in Lecture 2:

$$V^2(T_0, T_{h-1}) = \frac{1}{T_{h-1} - T_0} \int_{T_0}^{T_{h-1}} \sigma_h^2(t)dt, \quad h > 1.$$

In particular,

$$V^2(T_0, T_1) = \frac{1}{T_1 - T_0} \int_{T_0}^{T_1} \sigma_2^2(t)dt = \frac{1}{1} \int_{1y}^{2y} \sigma_2^2(t)dt =$$

$$= \int_{1y}^{2y} (\Phi_2 \psi_1)^2 dt = (\Phi_2 \psi_1)^2 = (0.11)^2$$

so that $V(T_0, T_1) = 0.11$.

$$V^2(T_0, T_2) = \frac{1}{T_2 - T_0} \int_{T_0}^{T_2} \sigma_3^2(t)dt = \frac{1}{2}(\int_{1y}^{3y} \sigma_3^2(t)dt) =$$

$$= \tfrac{1}{2}(\int_{1y}^{2y} \sigma_3^2(t)dt + \int_{2y}^{3y} \sigma_3^2(t)dt) =$$

$$= \tfrac{1}{2}\left(\int_{1y}^{2y} (\Phi_3\psi_2)^2 dt + \int_{2y}^{3y} (\Phi_3\psi_1)^2 dt\right) =$$

$$= \tfrac{1}{2}((\Phi_3\psi_2)^2 + (\Phi_3\psi_1)^2) = 0.01757$$

so that $V(T_0, T_2) = 0.132544$. Similarly,

$$V(T_0, T_3) = 0.1023, \;\; V(T_0, T_4) = 0.0866, \;\; V(T_0, T_5) = 0.1136$$

This completes the term structure of caplets volatilities in one year, i.e. at T_0.

$$0.11, \; 0.1325, \; 0.1023, \; 0.0866, \; 0.1136$$

Notice that it is enough to add the squares in each row of the "ziggurat" matrix up to the end, starting from the second column, then dividing by the number of years and taking square roots. Fast and simple in excel.

For the term structure in two years, i.e. at $T_1 = 2y$, the exercise is similar:

$$V^2(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \sigma_3^2(t) dt = \frac{1}{1} \int_{2y}^{3y} \sigma_3^2(t) dt =$$

$$= \int_{2y}^{3y} (\Phi_3\psi_1)^2 dt = (\Phi_3\psi_1)^2 = (0.12)^2$$

so that $V(T_1, T_2) = 0.12$.

$$V^2(T_1, T_3) = \frac{1}{T_3 - T_1}\int_{T_1}^{T_3} \sigma_4^2(t)dt = \frac{1}{2}(\int_{2y}^{4y} \sigma_4^2(t)dt) =$$

$$= \tfrac{1}{2}(\int_{2y}^{3y} \sigma_4^2(t)dt + \int_{3y}^{4y} \sigma_4^2(t)dt) =$$
$$= \tfrac{1}{2}(\int_{2y}^{3y} (\Phi_4\psi_2)^2 dt + \int_{3y}^{4y} (\Phi_4\psi_1)^2 dt) =$$
$$= \tfrac{1}{2}((\Phi_4\psi_2)^2 + (\Phi_4\psi_1)^2) = 0.009882$$

so that $V(T_1, T_3) = 0.0994$.

$$V^2(T_1, T_4) = \frac{1}{T_4 - T_1}\int_{T_1}^{T_4} \sigma_5^2(t)dt = \frac{1}{3}(\int_{2y}^{5y} \sigma_5^2(t)dt) =$$

$$= \frac{1}{3}(\int_{2y}^{3y} \sigma_5^2(t)dt + \int_{3y}^{4y} \sigma_5^2(t)dt + \int_{4y}^{5y} \sigma_5^2(t)dt) =$$
$$= \frac{1}{3}(\int_{2y}^{3y} (\Phi_5\psi_3)^2 dt + \int_{3y}^{4y} (\Phi_5\psi_2)^2 dt + \int_{4y}^{5y} (\Phi_5\psi_1)^2 dt) =$$
$$= \frac{1}{3}((\Phi_5\psi_3)^2 + (\Phi_5\psi_2)^2 + (\Phi_5\psi_1)^2) = 0.008277333$$

so that $V(T_1, T_4) = 0.09098$. And so on, with $V(T_1, T_5) = 0.11911$. The term structure of caplet volatilities in 2y (at time T_1) is

$$0.12,\ 0.0994,\ 0.09098,\ 0.11911.$$

Notice that it is enough to add the squares in each row of the "ziggurat" matrix up to the end, starting from the third column, then dividing by the number of years and taking square roots. Very fast in excel.

The term structure in three years is computed similarly:

$$0.09,\ 0.099363,\ 0.1251$$

and in Four years:

$$0.08,\ 0.1215$$

while, finally, in five years we just have $V(T_4, T_5) = 0.11$.

g) If $\beta = 0.2$, then

$$\rho_{1,2} = \rho_\infty + (1 - \rho_\infty)e^{-\beta|2-1|} = \rho_\infty + (1 - \rho_\infty)e^{-0.2}.$$

Since we know that $\rho_{1,2} = 0.9$, we solve

$$0.9 = \rho_\infty + (1 - \rho_\infty)e^{-0.2}$$

in ρ_∞. We obtain $\rho_\infty = 0.4483$.

h). To compute the terminal correlations in three years (time T_2) we use the formula

$$\text{Corr}(F_i(T_2), F_j(T_2)) = \rho_{i,j} \frac{\int_0^{T_2} \sigma_i(t)\sigma_j(t)\ dt}{\sqrt{\int_0^{T_2} \sigma_i^2(t)dt}\ \sqrt{\int_0^{T_2} \sigma_j^2(t)dt}}\ .$$

Let us compute for example $\text{Corr}(F_3(T_2), F_5(T_2))$, given by the above formula with $i = 3$ and $j = 5$. We need

$$\rho_{3,5} = \rho_\infty + (1-\rho_\infty)e^{-\beta|5-3|} = 0.4483 + (1-0.4483)0.6703$$

$$= 0.8181$$

Compute the numerator:

$$\int_0^{3y} \sigma_3(t)\sigma_5(t)\ dt = \int_0^{1y} \sigma_3(t)\sigma_5(t)\ dt +$$

$$+ \int_{1y}^{2y} \sigma_3(t)\sigma_5(t)\ dt + \int_{2y}^{3y} \sigma_3(t)\sigma_5(t)\ dt =$$

$$= \int_0^{1y} \Phi_3\psi_3\Phi_5\psi_5 dt + \int_{1y}^{2y} \Phi_3\psi_2\Phi_5\psi_4 dt +$$

$$+ \int_{2y}^{3y} \Phi_3\psi_1\Phi_5\psi_3 dt =$$

$$= \Phi_3\psi_3\Phi_5\psi_5 + \Phi_3\psi_2\Phi_5\psi_4 + \Phi_3\psi_1\Phi_5\psi_3 =$$

By substituting from the above ziggurat matrix we get $= 0.0311$. Again, it is faster to multiply the corresponding terms in the third and fifth rows of the ziggurat up to the third column included to get a faster evaluation:

$$0.144\ 0.064 + 0.144\ 0.072 + 0.12\ 0.096.$$

Let us compute the denominators. Notice that $\int_0^{T_2} \sigma_3^2(t)dt$ is three times the squared $T_2 - T_3$ caplet volatility, which is v_2^2 as computed in point e). We get $\int_0^{T_2} \sigma_3^2(t)dt = 3v_2^2 = 3\ 0.01862 = 0.05586$. Take square root to get 0.2363 as the first term in the denominator. For the second term, we need to compute

$$\int_0^{3y} \sigma_5^2(t)dt$$

It is enough to sum the squares of the fifth row (i.e. σ_5) in the ziggurat matrix of point e) up to the third column (i.e. up to three years) included. We get

$$\int_0^{3y} \sigma_5^2(t)dt = 0.064^2 + 0.072^2 + 0.096^2 = 0.018496.$$

Square root of this gives us 0.136. Our final calculation is thus

$$\text{Corr}(F_3(T_2), F_5(T_2)) = 0.8181 \frac{0.0311}{0.2363\ 0.136} = 0.7917$$

Once you have computed the other terminal correlations in three years and formed the matrix, you check whether the columns are decreasing when moving away from the diagonals. You also check there are no negative entries.

Exercise 2. a)

$$S_{1,4}(0) = \frac{P(0,T_1) - P(0,T_4)}{P(0,T_2) + P(0,T_3) + P(0,T_4)}$$

Similarly we obtain

$$S_{2,4}(0) = \frac{P(0,T_2) - P(0,T_4)}{P(0,T_3) + P(0,T_4)}$$

Substitution of the given values for the P's gives the result.

b) The annuity numeraire is $C_{2,4}(0) = P(0,T_3) + P(0,T_4)$. We also have

$$d_{1,2} = \frac{\ln(S_{2,4}(0)/K) \pm \frac{1}{2}\sigma^2 T_2}{\sigma\sqrt{T_2}} = \frac{\ln(0.04/0.05) \pm \frac{1}{2}(0.2)^2\ 3}{0.2\sqrt{3}}.$$

Substituting in Black's formula we have

$$C_{2,4}(0)[S_{2,4}(0)\Phi(d_1) - K\Phi(d_2)]$$

and inserting the given numbers we obtain the result.

c) Similar to b).

d) We know that

$$S_{2,4}(0) = w_3(0)F_3(0) + w_4(0)F_4(0),$$

where

$$w_3(0) = P(0,T_3)/(P(0,T_3) + P(0,T_4)),$$

$$w_4(0) = P(0,T_4)/(P(0,T_3) + P(0,T_4)).$$

Also,

$$F_3(0) = \frac{P(0,T_2)}{P(0,T_3)} - 1; \quad F_4(0) = \frac{P(0,T_3)}{P(0,T_4)} - 1.$$

Substitution of the given values for the P's gives the result.

Exercise 3 We change the numeraire from B to S:

$$E^{\boxed{B}}\left[\frac{\boxed{B(0)}}{\boxed{B(T)}} S_T^2\right] = E^{\boxed{S}}\left[\frac{\boxed{S_0}}{\boxed{S_T}} S_T^2\right]$$

$$= S_0 E^S[S_T].$$

Since we know that the dynamics of S under itself as numeraire is

$$dS_t = (r + \sigma^2)S_t dt + \sigma S_t dW_t^S,$$

we have

$$E^S[S_T] = S_0 e^{(r+\sigma^2)T},$$

so that the price of the quadratic payoff is

$$S_0 E^S[S_T] = S_0^2 e^{(r+\sigma^2)T}.$$

Notice that this price depends on the volatility