LECTURE 4

- **LECTURE 4**
- Pricing with the LIBOR model:
- Libor in Arrears;
- Constant Maturity Swaps (CMS);
- Ratchet caps and floors;
- Zero Coupon Swaptions.

LIBOR model pricing: General approximation

Freeze part of the drift of the LIBOR dynamics so as to obtain a "multi-dimensional" geometric Brownian motion. This was done earlier to derive approximated formulas for swap volatilities and terminal correlations. Recall: under the T_i-forward-adjusted measure Q^i we have the exact dynamics:

$$dF_k(t) = \mu_{i,k}(t)F_k(t)\,dt + \sigma_k(t)F_k(t)\,dZ_k^i(t)\,,$$

where $\mu_{i,k}(t) := \sigma_k(t)\mu_i^k(t)$ for $i < k$, $\mu_{i,i}(t) := 0$ and $\mu_{i,k}(t) := -\sigma_k(t)\mu_k^i(t)$ for $i > k$ (see Lecture 1). To sum up:

$$\begin{aligned}
\mu_{i,k}(t) &:= -\sigma_k(t)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}\,, \quad k < i \\
\mu_{i,k}(t) &:= 0, \quad k = i \\
\mu_{i,k}(t) &:= \sigma_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}\,, \quad k > i.
\end{aligned}$$

The distributions or statistical laws of the F_k under Q^i are unknown for $i \neq k$. This is a problem, because prices are expectations under pricing measures, and if we do not know the laws of the random variables we cannot compute the expectations analytically. We are forced to resort to numerical methods. Can we escape this situation in some cases?

LIBOR model pricing: General approximation

$$dF_k(t) = \mu_{i,k}(t) F_k(t)\, dt + \sigma_k(t) F_k(t)\, dZ_k^i(t)\,,$$

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\mu_{i,k}(t) &:= 0, \quad k = i\\
\mu_{i,k}(t) &:= \sigma_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}\,, \quad k > i.
\end{aligned}$$

Consider the approximated **lognormal** dynamics:

$$dF_k(t) = \bar{\mu}_{i,k}(t) F_k(t)\, dt + \sigma_k(t) F_k(t)\, dZ_k^i(t)\,,$$

$$\begin{aligned}
\bar{\mu}_{i,k}(t) &:= -\sigma_k(t) \sum_{j=k+1}^{i} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(\mathbf{0})}{1+\tau_j F_j(\mathbf{0})}\,, \quad k < i\\
\bar{\mu}_{i,k}(t) &:= 0, \quad k = i\\
\bar{\mu}_{i,k}(t) &:= \sigma_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(\mathbf{0})}{1+\tau_j F_j(\mathbf{0})}\,, \quad k > i.
\end{aligned}$$

LIBOR model pricing: General approximation

Consider the approximated **lognormal** dynamics:

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$$\begin{aligned}\bar{\mu}_{i,k}(t) &:= -\sigma_k(t)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(\mathbf{0})}{1+\tau_jF_j(\mathbf{0})}, \quad k<i\\ \bar{\mu}_{i,k}(t) &:= 0, \quad k=i\\ \bar{\mu}_{i,k}(t) &:= \sigma_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(\mathbf{0})}{1+\tau_jF_j(\mathbf{0})}, \quad k>i.\end{aligned}$$

This dynamics gives access, in some cases, to a number of techniques which have been developed for the basic Black and Scholes setup, for example, in equity and FX markets. Moreover, this “freezing-part-of-the-drift” technique can be combined with drift interpolation so as to allow for rates that are not in the fundamental (spanning) family $T_0, T_1, \ldots, T_M$ corresponding to the particular model being implemented. Finally, even resorting to MC allows now for a “one-shot” propagation of the dynamics with no infra-discretization, thus reducing memory requirements and simulation time. A similar idea may work also in some smile extensions.

Libor in Arrears (In Advance Swaps)

An in-advance swap (or LIBOR in arrears) is an IRS that resets at dates $T_{\alpha+1}, \dots, T_\beta$ and pays at the same dates, with unit notional amount and with fixed-leg rate K.

With respect to standard swaps, the LIBOR payments are "in arrears", since the libor pays immediately when it resets, and not one period later.

More precisely, the discounted payoff of an in-advance swap (of "payer" type) can be expressed via

$$\sum_{i=\alpha+1}^{\beta} \frac{B(0)}{B(T_i)} \tau_{i+1} (L(T_i, T_{i+1}) - K) =$$

$$= \sum_{i=\alpha+1}^{\beta} \frac{B(0)}{B(T_i)} \tau_{i+1} (F_{i+1}(T_i) - K).$$

The value of such a contract is, therefore,

$$\textbf{IAS} = E^B \left[\sum_{i=\alpha+1}^{\beta} \frac{B(0)}{B(T_i)} \tau_{i+1} (F_{i+1}(T_i) - K) \right].$$

Libor in Arrears (In Advance Swaps)

Before calculating the expectations, it is convenient to make some adjustments. We shall use the following identity (obtained easily via iterated conditioning):

$$E^B\left[X_T\frac{B(0)}{B(T)}\right] = E^B\left[\frac{P(T,S)}{P(T,S)}\frac{B(0)}{B(T)}X_T\right] =$$

$$= E^B\left\{\frac{1}{P(T,S)}X_T\frac{B(0)}{B(T)}E^B\left[\frac{B(T)}{B(S)}1\middle|\text{Info}_T\right]\right\} =$$

$$= E^B\left\{E^B\left[\frac{1}{P(T,S)}X_T\frac{B(0)}{B(T)}\frac{B(T)}{B(S)}1\middle|\text{Info}_T\right]\right\} =$$

$$= E^B\left\{E^B\left[\frac{1}{P(T,S)}X_T\frac{B(0)}{B(S)}\middle|\text{Info}_T\right]\right\} =$$

$$= E^B\left[\frac{1}{P(T,S)}X_T\frac{B(0)}{B(S)}\right] \quad \text{so that}$$

$$\boxed{E^B\left[X_T\frac{B(0)}{B(T)}\right] = E^B\left[\frac{X_T\frac{B(0)}{B(S)}}{P(T,S)}\right]} \quad \text{for all } 0 < T < S,$$

where X is a T-measurable random variable, known from Info_T

To value the above contract, notice that

$$E\left\{\sum_{i=\alpha+1}^{\beta}\frac{B(0)}{B(T_i)}\tau_{i+1}(F_{i+1}(T_i)-K)\right\}$$

$$= E\left\{\sum_{i=\alpha+1}^{\beta}D(0,T_i)\left[\frac{1}{P(T_i,T_{i+1})}-(1+\tau_{i+1}K)\right]\right\} = \ldots$$

Now use our previous result with $T = T_i$, $S = T_{i+1}$, $X_T = 1/P(T_i,T_{i+1})$ to get

$$E\left[\frac{B(0)}{B(T_i)}\frac{1}{P(T_i,T_{i+1})}\right] = E\left[\frac{\frac{1}{P(T_i,T_{i+1})}\frac{B(0)}{B(T_{i+1})}}{P(T_i,T_{i+1})}\right] =$$

$$= E\left[\frac{1}{P(T_i,T_{i+1})^2}\frac{B(0)}{B(T_{i+1})}\right]$$

and substitute:

$$= E\left\{\sum_{i=\alpha+1}^{\beta}\left[\frac{B(0)}{B(T_{i+1})}\frac{1}{P(T_i,T_{i+1})^2}-\frac{B(0)}{B(T_i)}(1+\tau_{i+1}K)\right]\right\}$$

$$= E \sum_{i=\alpha+1}^{\beta} \left[\frac{B(0)}{B(T_{i+1})} \frac{1}{P(T_i, T_{i+1})^2} - \frac{B(0)}{B(T_i)}(1 + \tau_{i+1} K) \right]$$

$$= \sum_{i=\alpha+1}^{\beta} E^{\boxed{B}} \left[\frac{\boxed{B(0)}}{\boxed{B(T_{i+1})}} \frac{1}{P(T_i, T_{i+1})^2} \right]$$

$$- \sum_{i=\alpha+1}^{\beta} E^{B} \left[\frac{B(0)}{B(T_i)}(1 + \tau_{i+1} K) \right]$$

$$= \sum_{i=\alpha+1}^{\beta} \boxed{P(0, T_{i+1})} E^{\boxed{i+1}} \left[\frac{1}{\boxed{P(T_{i+1}, T_{i+1})}} \frac{1}{P(T_i, T_{i+1})^2} \right]$$

$$- \sum_{i=\alpha+1}^{\beta} P(0, T_i)(1 + \tau_{i+1} K)$$

$$= \sum_{i=\alpha+1}^{\beta} P(0, T_{i+1}) E^{i+1} \left[(1 + \tau_{i+1} F_{i+1}(T_i))^2 \right]$$

$$- \sum_{i=\alpha+1}^{\beta} P(0, T_i)(1 + \tau_{i+1} K).$$

Computing the expected value is an easy task, since we know that, under Q^{i+1}, F_{i+1} has the driftless (martingale) lognormal dynamics

$$dF_{i+1}(t) = \sigma_{i+1}(t) F_{i+1}(t) dZ_{i+1}(t) \ ,$$

so that (Ito formula $\phi(F) = F^2,\ \phi'(F) = 2F,\ \phi''(F) = 2$),

$$\begin{aligned} dF_{i+1}^2(t) &= 2F_{i+1}(t)dF_{i+1}(t) + \tfrac{1}{2}2dF_{i+1}(t)dF_{i+1}(t) \\ &= \sigma_{i+1}(t)^2 F_{i+1}^2(t)dt + 2\sigma_{i+1}(t)F_{i+1}^2(t)dZ_{i+1}(t) \ , \end{aligned}$$

so that we still have a geometric brownian motion for F^2:

$$dF_{i+1}^2(t) = \sigma_{i+1}(t)^2 F_{i+1}^2(t)dt + 2\sigma_{i+1}(t)F_{i+1}^2(t)dZ_{i+1}(t) \ ,$$

and the mean of this process is known to be

$$\begin{aligned} E^{i+1}\left(F_{i+1}^2(T_i)\right) &= F_{i+1}^2(0)\exp\left[\int_0^{T_i} \sigma_{i+1}^2(t)dt\right] \\ &= F_{i+1}^2(0)\exp(T_i v_i^2) \end{aligned}$$

where the v's have been defined in Lecture 2 and are caplet volatilities for $T_i - T_{i+1}$.

By expanding the square and substituting we obtain

$$\begin{aligned} \textbf{IAS} &= \sum_{i=\alpha+1}^{\beta} \{ P(0,T_{i+1}) \left[1 + 2\tau_{i+1} F_{i+1}(0) + \right. \\ &+ \left. \tau_{i+1}^2 F_{i+1}^2(0) \exp(v_i^2 T_i) \right] - (1 + \tau_{i+1} K) P(0,T_i) \}. \end{aligned}$$

Contrary to the plain-vanilla case, this price depends on the volatility of forward rates through the caplet volatilities v. Notice however that correlations between different rates are not involved in this product, as one expects from the additive and "one-rate-per-time" nature of the payoff.

Ratchet Caps and Floors

A ratchet cap is a cap where the strike is updated at each caplet reset, based on the previous realization of the relevant interest rate.
A simple ratchet cap first resetting at T_α and paying at $T_{\alpha+1}, \ldots, T_\beta$ pays the following discounted payoff:

$$\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i \left[L(T_{i-1},T_i) - (L(T_{i-2},T_{i-1}) + X)\right]^+ ,$$

Notice that if we set $K_i := L(T_{i-2},T_{i-1}) + X$ for all i's this is a set of caplets with (random) strikes K_i.

X is a margin, which can be either positive or negative.

A **sticky** ratchet cap is instead given by:

$$\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i \left[L(T_{i-1},T_i) - X_i\right]^+ ,$$

$$X_i = \max\left(L(T_{i-2},T_{i-1}) \pm \bar{X}, X_{i-1} \pm \bar{X}\right),$$

$$X_\alpha := L(T_{\alpha-1},T_\alpha).$$

There are versions with "min" replacing "max" in the X_i's definition. The quantity $\bar{X}$ is a spread that can be positive or negative.

Ratchet caps and floors

In general a sticky ratchet cap has to be valued through Monte Carlo simulation. We have

$$E\{\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i\,[L(T_{i-1},T_i)-X_i]^+\}$$

$$= P(0,T_\beta)\sum_{i=\alpha+1}^{\beta}\tau_i E^\beta\left\{\frac{[L(T_{i-1},T_i)-X_i]^+}{P(T_i,T_\beta)}\right\}.$$

Since the Q^β forward-rate dynamics of $F_{\beta(t)}(t),\ldots,F_\beta(t)$ can be discretized via the usual scheme, Monte Carlo pricing can be carried out in the usual manner. We can use also the lognormal frozen-drift approximation to implement a faster MC simulation.

However, for the non-sticky ratchet cap payoff we may investigate possible analytical approximations based on the usual "freezing the drift" technique for the LIBOR market model.

Non-Sticky Ratchets: Analytical approximation

All we need to compute is the expectation

$$E\{D(0,T_i)\,[L(T_{i-1},T_i) - (L(T_{i-2},T_{i-1}) + X)]^+\}$$

$$= P(0,T_i)E^i\{[F_i(T_{i-1}) - F_{i-1}(T_{i-2}) - X]^+\} =: P(0,T_i)m_i$$

and then add terms. In the above expectation, the rates evolve as follows under the measure Q^i: $dF_i(t) = \sigma_i(t)F_i(t)dZ_i(t)$,

$$dF_{i-1}(t) = -\frac{\rho_{i-1,i}\tau_i\sigma_i(t)F_i(t)}{1+\tau_iF_i(t)}F_{i-1}(t)dt + \sigma_{i-1}(t)F_{i-1}(t)dZ_{i-1}(t)$$

As usual, in such dynamics we do not know the distribution of $F_{i-1}(t)$. But, since F_{i-1} and F_i's reset times are adjacent, we may freeze the drift in F_{i-1} and be rather confident on the resulting approximations. We thus replace the second SDE by

$$\begin{aligned} dF_{i-1}(t) &= \bar{\mu}(t)F_{i-1}(t)dt + \sigma_{i-1}(t)F_{i-1}(t)dZ_{i-1}(t), \\ \bar{\mu}(t) &:= -\frac{\rho_{i-1,i}\tau_i\sigma_i(t)F_i(0)}{1+\tau_iF_i(0)}. \end{aligned}$$

Now both F_{i-1} and F_i follow (correlated) geometric Brownian motions as in the Black and Scholes model.

Non-Sticky Ratchets: Analytical approximation. Spread option

Now consider the case $X > 0$.

If we set $S_1 := F_i$, $S_2 := F_{i-1}$, $r - q_1 := 0$, $r - q_2 := \bar{\mu}(t)1\{t < T_{i-2}\}$, $\sigma_1 := \sigma_i(t)$, $\sigma_2 := \sigma_{i-1}(t)1\{t < T_{i-2}\}$, $a = 1$, $b = -1$, and $\omega = 1$, we may view our dynamics as the two-dimensional Black Scholes dynamics $d[S_1, S_2]$ and our payoff as a spread option payoff, by slightly adjusting to the fact that no discounting should occur in our case.

Consider two assets whose prices S_1 and S_2 evolve, under the risk neutral measure, according to

$$dS_1(t) = S_1(t)[(r - q_1)dt + \sigma_1 dW_1^Q(t)], \quad S_1(0) = s_1,$$
$$dS_2(t) = S_2(t)[(r - q_2)dt + \sigma_2 dW_2^Q(t)], \quad S_2(0) = s_2,$$

where W_1^Q and W_2^Q are Brownian motions under Q with instantaneous correlation ρ.
Fix a maturity T, a positive real number a, a negative real number b, a strike price K. The spread-option payoff at time T is then defined by

$$H = (awS_1(T) + bwS_2(T) - wK)^+, \qquad (9)$$

where $w = 1$ for a call and $w = -1$ for a put.

Non-Sticky Ratchets: Analytical approximation. Spread option

Price of the spread option:

$$\pi_t = e^{-r(T-t)} E_t^Q \left\{ (awS_1(T) + bwS_2(T) - wK)^+ \right\},$$

A pseudo-analytical formula can be derived. The unique arbitrage-free price is

$$\pi_t = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} f(v) dv,$$

where

$$f(v) = awS_1(t) \exp\left[-q_1\tau - \frac{1}{2}\rho^2\sigma_1^2\tau + \rho\sigma_1\sqrt{\tau}v \right] \cdot$$

$$\cdot \Phi\left(w \frac{\ln \frac{aS_1(t)}{h(v)} + [\mu_1 + (\frac{1}{2} - \rho^2)\sigma_1^2]\tau + \rho\sigma_1\sqrt{\tau}v}{\sigma_1\sqrt{\tau}\sqrt{1-\rho^2}} \right)$$

$$-wh(v)e^{-r\tau}\Phi\left(w \frac{\ln \frac{aS_1(t)}{h(v)} + (\mu_1 - \frac{1}{2}\sigma_1^2)\tau + \rho\sigma_1\sqrt{\tau}v}{\sigma_1\sqrt{\tau}\sqrt{1-\rho^2}} \right)$$

and

$$h(v) = K - bS_2(t)e^{(\mu_2 - \frac{1}{2}\sigma_2^2)\tau + \sigma_2\sqrt{\tau}v}, \ \ \mu_{1,2} = r - q_{1,2}, \ \tau = T - t.$$

proof based on standard bivariate Gaussian variables comp.

Non-Sticky Ratchets: Analytical approximation.

If $X < 0$ we just switch the definitions of S_1 and S_2 above, $S_1 := F_{i-1}$, $S_2 = F_i$ etc., and then take $\omega = -1$. In the calculations below we assume $X > 0$.

As a matter of fact, our coefficients here are time-dependent, but this does not change substantially the derivation. It follows that our expected value

$$m_i := E^i\{[F_i(T_{i-1}) - F_{i-1}(T_{i-2}) - X]^+\}$$

is given by our formula above for the spread option when taking into account the above substitutions, i.e. one needs to apply said formula with $a = 1,\ \omega = 1,\ t = 0$,

$$S_1(t) = F_i(0),\ q_1\tau = 0,\ q_2\tau = -\int_0^{T_{i-2}} \bar{\mu}(u)du,$$

$$r = 0,\quad \sigma_1^2\tau = \int_0^{T_{i-1}} \sigma_i^2(u)du,\ \sigma_2^2\tau = \int_0^{T_{i-2}} \sigma_{i-1}^2(u)du,$$

$$\rho = \rho_{i-1,i},\quad K = X.$$

Once we have the m_i's, our ratchet price is given by

$$\sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i m_i.$$

Non-Sticky Ratchets: Analytical approximation. Case $X = 0$.

The above price depends on a one-dimensional numerical integration. There is a case, though, where this is not necessary. Indeed, if $X = 0$, we obtain a special ratchet cap that, under the lognormal assumption, we may value analytically through the **Margrabe formula for the option exchanging one asset for another**.

We map the ratchet payoff terms

$$E^i\left[(F_i(T_{i-1}) - F_{i-1}(T_{i-2}))^+\right]$$

into equity payoffs

$$H = (S_1(T) - S_2(T))^+$$

This payoff is the so called "option to exchange one asset (S_1) for another (S_2)". Indeed, if we hold S_2, when we are at T the option pays

$$(S_1(T) - S_2(T))^+ = \max(S_1(T) - S_2(T), 0) = S_1(T) - S_2(T)$$

if $S_1(T) > S_2(T)$, and 0 otherwise. Recall we are holding S_2. By getting the option payoff in this case where $S_1(T) > S_2(T)$, we get a total of

$$S_2(T) + (S_1(T) - S_2(T)) = S_1(T).$$

Non-Sticky Ratchets: Analytical approximation. Case $X = 0$.

So, when holding S_2, if $S_1(T) > S_2(T)$ by means of the option we end up with S_1, so we have exchanged S_2 with the more valuable S_1. On the contrary, if $S_1(T) < S_2(T)$, the option expires worthless and there is no exchange, so we keep the more valuable S_2.

This means that the exchange is a right but no obligation, since it occurs only when it favors us.

This is then indeed an option to exchange one asset for another. In the market this kind of option is priced with Margrabe's formula, which we derive below by using the change of numeraire technique.

Non-Sticky Ratchets: Analytical approximation. Case $X = 0$.

We derive a formula now for

$$E^B\left[\frac{B(0)}{B(T)}(S_1(T) - S_2(T))^+\right] = E^{S_1}\left[\frac{S_1(0)}{S_1(T)}(S_1(T) - S_2(T))^+\right] =$$

$$= S_1(0)E^{S_1}\left[\left(1 - \frac{S_2(T)}{S_1(T)}\right)^+\right] = S_1(0)E^{S_1}\left[(1 - Y(T))^+\right]$$

where $Y_t = S_2(t)e^{-\int_t^T q_2(s)ds}/S_1(t)$. Note that we took S_1 as numeraire, assuming $q_1 = 0$, since the numeraire has to be a positive non-dividend paying asset ($q_1 = 0$). Notice also that in the numerator, to have the price of a tradable asset, we got rid of the dividend by inserting the forward price

$$E_t^B\left[\frac{B(t)}{B(T)}S_2(T)\right] = S_2(t)e^{-\int_t^T q_2(s)ds}$$

(without dividends the forward price would be $S_2(t)$ itself).

Now we need to derive the dynamics of Y_t under the S_1 measure. We know this is a martingale, since Y_t is a ratio between a tradable asset and our numeraire S_1, so that by FACT ONE on the change of numeraraire (first lecture) we have that Y_t is a martingale (=zero drift).

Compute then, first under Q^B:

$$dY_t = d\left(\frac{S_2(t)e^{-\int_t^T q_2(s)ds}}{S_1(t)}\right) = d\left(e^{-\int_t^T q_2(s)ds}\frac{S_2(t)}{S_1(t)}\right) =$$

First notice that the first term would only give a "dt" contribution when differentiated. Then we compute directly

$$= e^{-\int_t^T q_2(s)ds} d\left(\frac{S_2(t)}{S_1(t)}\right) + (\dots)dt =$$

$$= e^{-\int_t^T q_2(s)ds}\left[\frac{1}{S_1(t)}d\left(S_2(t)\right) + S_2(t)d\left(\frac{1}{S_1(t)}\right) + \right.$$
$$\left. + dS_2(t)\ d\left(\frac{1}{S_1(t)}\right)\right] + (\dots)dt =$$
$$= e^{-\int_t^T q_2(s)ds}\left[\frac{1}{S_1(t)}d\left(S_2(t)\right) + S_2(t)d\left(\frac{1}{S_1(t)}\right)\right] + (\dots)dt =$$
$$= e^{-\int_t^T q_2(s)ds}\left\{\frac{1}{S_1(t)}S_2(t)[(r-q_2)dt + \sigma_2 dW_2^B] + \right.$$
$$\left. + S_2(t)d\left(\frac{1}{S_1(t)}\right)\right\} + (\dots)dt = \dots \rightarrow$$

Since (Ito $\phi(S) = \frac{1}{S}$, $\phi'(S) = -\frac{1}{S^2}$, $\phi''(S) = 2/(S^3)$)

$$d\left(\frac{1}{S_1(t)}\right) = \frac{-1}{S_1^2}dS_1 + \frac{1}{2}\frac{2}{S_1^3}dS_1 dS_1 = -\frac{1}{S_1}[r dt + \sigma_1 dW_1^B] + (\dots)dt,$$

by substituting we obtain

$$\rightarrow \ldots = e^{-\int_t^T q_2(s)ds} \left[\frac{1}{S_1(t)} S_2(t)\sigma_2 dW_2^B + S_2(t) \left(-\frac{1}{S_1} \sigma_1 dW_1^B \right) \right]$$

$$+(\ldots)dt =$$

Recalling that $Y_t = S_2 e^{-\int q_2}/S_1$, we may then write

$$dY_t = -Y_t \sigma_1 dW_1^B + Y_t \sigma_2 dW_2^B + (\ldots)dt$$

Now, if we change numeraire, the diffusion part does not change. Since we already know that under the S_1 measure Y is a martingale, this means that the equation of Y under S_1 will have the same diffusion parts and zero drift. We get

$$dY_t = -Y_t \sigma_1 dW_1^{S_1} + Y_t \sigma_2 dW_2^{S_1}$$

or also

$$dY_t = Y_t(-\sigma_1 dW_1^{S_1} + \sigma_2 dW_2^{S_1})$$

From the point of view of the law, this process is the same as a process with a single brownian motion

$$dY_t = Y_t(\sigma_0 dW_0^{S_1})$$

provided that

$$\mathrm{Var}(\sigma_2 dW_2^{S_1} - \sigma_1 dW_1^{S_1}) = \mathrm{Var}(\sigma_0 dW_0^{S_1})$$

This equation reads

$$\text{Var}(\sigma_2 dW_2^{S_1} - \sigma_1 dW_1^{S_1}) = \sigma_1^2 dt + \sigma_2^2 dt - 2\rho\sigma_1\sigma_2 dt$$

and since

$$\text{Var}(\sigma_0 dW_0^{S_1}) = \sigma_0^2 dt$$

we have

$$\sigma_0^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$$

Let us now go back to

$$S_1(0)E^{S_1}\left[(1 - Y(T))^+\right]$$

with the dynamics

$$dY_t = Y_t(\sigma_0 dW_0^{S_1})$$

This is a put option with strike 1 for which we get the formula

$$S_1(0)[\Phi(-d_2) - Y(0)\Phi(-d_1)]$$

with

$$d_{1,2} = \frac{\ln(Y(0)/1) \pm \frac{1}{2}\int_0^T \sigma_0^2(t)dt}{\left(\int_0^T \sigma_0^2(t)dt\right)^{\frac{1}{2}}}$$

Recalling the expressions for Y and σ_0 we get

$$[S_1(0)\Phi(-d_2) - S_2(0)e^{-\int_0^T q_2(t)dt}\Phi(-d_1)]$$

$$d_{1,2} = \frac{\ln(S_2(0)/S_1(0)) - \int_0^T q_2(t)dt \pm \frac{1}{2}\int_0^T [...]dt}{\left(\int_0^T [\sigma_1^2(t) + \sigma_2^2(t) - 2\rho\sigma_1(t)\sigma_2(t)]dt\right)^{\frac{1}{2}}}$$

As before, set

$$S_1(t) = F_i(0),\ q_1 = 0,\ \int_0^T q_2(t)dt = -\int_0^{T_{i-2}} \bar{\mu}(u)du,$$

$$r = 0,\ \int_0^T \sigma_1^2(t)dt = \int_0^{T_{i-1}} \sigma_i^2(t)dt,$$

$$\int_0^T \sigma_2^2(t)dt = \int_0^{T_{i-2}} \sigma_{i-1}^2(t)dt,\ \rho = \rho_{i-1,i}$$

to get:

Non-Sticky Ratchets: Analytical approximation. Case $X = 0$.

$$E\left\{\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i \left[L(T_{i-1},T_i) - L(T_{i-2},T_{i-1})\right]^+\right\}$$

$$= E\left\{\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i \left[F_i(T_{i-1}) - F_{i-1}(T_{i-2})\right]^+\right\}$$

$$\approx \sum_{i=\alpha+1}^{\beta} \tau_i \; P(0,T_i)\left[F_i(0)\Phi(d_1^i) - F_{i-1}(0)\exp\left(\int_0^{T_{i-2}} \bar{\mu}(u)du\right)\Phi(d_2^i)\right],$$

$$d_{1,2}^i = \frac{\ln(F_i(0)/F_{i-1}(0)) - \int_0^{T_{i-2}} \bar{\mu}(u)du}{R_i} \pm \tfrac{1}{2}R_i,$$

$$R_i = \left(\int_0^{T_{i-1}} \sigma_i^2(u)du + \int_0^{T_{i-2}}(\sigma_{i-1}^2(u) - 2\rho_{i-1,i}\sigma_{i-1}(u)\sigma_i(u))du\right)^{\frac{1}{2}}$$

In this section we dealt with ratchet caps. The treatment of ratchet floors is analogous.

Zero Coupon Swaption

A payer (receiver) zero-coupon swaption is a contract giving the right to enter a payer (receiver) zero-coupon IRS at a future time. A zero-coupon IRS is an IRS where a single fixed payment is due at the unique (final) payment date T_β for the fixed leg in exchange for a stream of usual floating payments $\tau_i L(T_{i-1}, T_i)$ at times T_i in $T_{\alpha+1}, T_{\alpha+2}, \ldots, T_\beta$ (usual floating leg). In formulas, the discounted payoff of a payer zero-coupon IRS is, at time $t \leq T_\alpha$:

$$\frac{B(t)}{B(T_\alpha)}\left[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta}K\right],$$

where $\tau_{\alpha,\beta}$ is the year fraction between T_α and T_β. The analogous payoff for a receiver zero-coupon IRS is obviously given by the opposite quantity.

Taking risk-neutral expectation, we obtain easily the contract value as

$$P(t, T_\alpha) - P(t, T_\beta) - \tau_{\alpha,\beta}KP(t, T_\beta),$$

which is the typical value of a floating leg minus the value of a fixed leg with a single final payment.

Zero Coupon Swaption

The value of K that renders the contract fair is obtained by equating to zero the above value. $K = F(t; T_\alpha, T_\beta)$. Indeed, the value of the swap is independent of the number of payments on the floating leg, since the floating leg always values at par, no matter the number of payments. Therefore, we might as well have taken a floating leg paying only in T_β the amount $\tau_{\alpha,\beta} L(T_\alpha, T_\beta)$. This would have given us again a standard swaption, standard in the sense that the two legs of the underlying IRS have the same payment dates (collapsing to T_β) and the unique reset date T_α. In such a one-payment case, the swap rate collapses to a forward rate, so that we should not be surprised to find out that the forward swap rate in this particular case is simply a forward rate.

Zero Coupon Swaption

An option to enter a payer zero-coupon IRS is a payer zero-coupon swaption, and the related payoff is

$$\frac{B(t)}{B(T_\alpha)}[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta}K]^+,$$

or, equivalently, by expressing the F's in terms of discount factors,

$$\frac{B(t)}{B(T_\alpha)}\left[1 - P(T_\alpha, T_\beta) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta}K\right]^+,$$

which in turn can be written as

$$\frac{B(t)}{B(T_\alpha)}\tau_{\alpha,\beta}P(T_\alpha, T_\beta)\left[F(T_\alpha; T_\alpha, T_\beta) - K\right]^+.$$

Zero Coupon Swaption

$$\frac{B(t)}{B(T_\alpha)}\tau_{\alpha,\beta}P(T_\alpha,T_\beta)\left[F(T_\alpha;T_\alpha,T_\beta)-K\right]^+.$$

Notice that, from the point of view of the payoff structure, this is merely a caplet. As such, it can be priced easily through Black's formula for caplets. The problem, however, is that such a formula requires the integrated percentage variance (volatility) of the forward rate $F(\cdot;T_\alpha,T_\beta)$, which is a forward rate over a non-standard period. Indeed, $F(\cdot;T_\alpha,T_\beta)$ is not in our usual family of spanning forward rates, unless we are in the trivial case $\beta=\alpha+1$. Therefore, since the market provides us (through standard caps and swaptions) with volatility data for standard forward rates, we need a formula for deriving the integrated percentage volatility of the forward rate $F(\cdot;T_\alpha,T_\beta)$ from volatility data of the standard forward rates $F_{\alpha+1},\ldots,F_\beta$. The reasoning is once again based on the "freezing the drift" procedure, leading to an approximately lognormal dynamics for our standard forward rates.

Zero Coupon Swaption

Denote for simplicity $F(t) := F(t; T_\alpha, T_\beta)$ and $\tau := \tau_{\alpha,\beta}$.

We begin by noticing that, through straightforward algebra, we have (write everything in terms of discount factors to check)

$$1 + \tau F(t) = \prod_{j=\alpha+1}^{\beta} (1 + \tau_j F_j(t)).$$

It follows that

$$\ln(1 + \tau F(t)) = \sum_{j=\alpha+1}^{\beta} \ln(1 + \tau_j F_j(t)),$$

so that $d \ln(1 + \tau F(t)) =$

$$= \sum_{j=\alpha+1}^{\beta} d \ln(1 + \tau_j F_j(t)) = \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1 + \tau_j F_j(t)} + (\ldots)dt.$$

Since $dF(t) = \dfrac{1 + \tau F(t)}{\tau} d \ln(1 + \tau F(t)) + (\ldots)dt$, we obtain from the above expression

$$dF(t) = \frac{1 + \tau F(t)}{\tau} \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1 + \tau_j F_j(t)} + (\ldots)dt.$$

Zero Coupon Swaption

$$dF(t) = \frac{1+\tau F(t)}{\tau} \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1+\tau_j F_j(t)} + (\ldots)dt.$$

Take variance (conditional on t) on both sides:

$$\mathrm{Var}\left(\frac{dF(t)}{F(t)}\right) = \left[\frac{1+\tau F(t)}{\tau F(t)}\right]^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{i,j} \sigma_i(t)\sigma_j(t) F_i(t) F_j(t)}{(1+\tau_i F_i(t))(1+\tau_j F_j(t))} dt.$$

Freeze all $t's$ to 0 except for the σ's, and integrate over $[0, T_\alpha]$:
$(v^{zc}_{\alpha,\beta})^2 := (1/T_\alpha)\times$

$$\left[\frac{1+\tau F(0)}{\tau F(0)}\right]^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{i,j} F_i(0) F_j(0)}{(1+\tau_i F_i(0))(1+\tau_j F_j(0))} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t) dt.$$

To price the zero-coupon swaption it is then enough to put this quantity into the related Black's Caplet formula:

$$\textbf{ZCPS} = \tau P(0,T_\beta)[F(0)\Phi(d_1(F(0),K,v^{zc}_{\alpha,\beta}))$$
$$-K\Phi(d_2(F(0),K,v^{zc}_{\alpha,\beta}))].$$

Zero Coupon Swaption

We have checked the accuracy of this formula against the usual Monte Carlo pricing based on the exact dynamics of the forward rates. In the tests all swaptions are at-the-money. We have done this under a number of situations , corresponding to possible modifications of the data coming from a standard calibrations of the LIBOR model to at-the-money swaptions data.

All cases show the formula to be sufficiently accurate for practical purposes.

When using the formula we notice that the at-the-money standard swaption has always a lower volatility (and hence price) than the corresponding at-the-money zero-coupon swaption. We may wonder whether this is a general feature. Indeed, we have the following.

Zero Coupon Swaption

Comparison between zero-coupon swaptions and corresponding standard swaptions: A first remark is due for a comparison between the zero-coupon swaption volatility $v^{zc}_{\alpha,\beta}$ and the corresponding European-swaption approximation $v^{\mathsf{LFM}}_{\alpha,\beta}$. If we rewrite the latter as

$$T_\alpha (v^{\mathsf{LFM}}_{\alpha,\beta})^2 = \sum_{i,j=\alpha+1}^{\beta} \rho_{i,j} \lambda_i \lambda_j \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t) dt, \;\; \lambda_i = \frac{w_i(0) F_i(0)}{S_{\alpha,\beta}(0)},$$

it is easy to check that

$$T_\alpha (v^{zc}_{\alpha,\beta})^2 = \sum_{i,j=\alpha+1}^{\beta} \rho_{i,j} \mu_i \mu_j \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t) dt,$$

where

$$\mu_i = \frac{P(0,T_\alpha)}{P(0,T_i)} \lambda_i \geq \lambda_i,$$

the discrepancy increasing with the payment index i. It follows that, for positive correlations, the zero-coupon swaption volatility is always larger than the corresponding plain vanilla swaption volatility, the difference increasing with the tenor $T_\beta - T_\alpha$, for each given T_α.

Constant Maturity Swaps (CMS's)

A constant-maturity swap is a financial product structured as follows. We assume a unit nominal amount. Let us denote by $\{T_0, \ldots, T_n\}$ a set of payment dates at which coupons are to be paid. At time T_{i-1} (in some variants at time T_i), $i \geq 1$, institution A pays to B the c-year swap rate resetting at time T_{i-1} in exchange for a fixed rate K. Formally, at time T_{i-1} institution A pays to B

$$S_{i-1,i-1+c}(T_{i-1})\,\tau_i\,,$$

instead of

$$L(T_{i-1}, T_i)\tau_i = F_i(T_{i-1})\,\tau_i\,,$$

as would be natural (standard Interest Rate Swap with model independent valuation, see Lecture 1).

Constant Maturity Swaps (CMS's)

The net value of the contract to B at time 0 is

$$E^B\left(\sum_{i=1}^{n}\frac{B(0)}{B(T_{i-1})}(S_{i-1,i-1+c}(T_{i-1})-K)\tau_i\right)$$

$$=\sum_{i=1}^{n}\tau_i P(0,T_{i-1})\left[E^{i-1}\left(S_{i-1,i-1+c}(T_{i-1})\right)-K\right]$$

$$=\sum_{i=1}^{n}\tau_i E^{\boxed{B}}\left[\frac{\boxed{B(0)}}{\boxed{B(T_{i-1})}}S_{i-1,i-1+c}(T_{i-1})\right]$$

$$-K\sum_{i=1}^{n}\tau_i P(0,T_{i-1})$$

We can change numeraire in two ways:

$$1: \quad \rightarrow=\sum_{i=1}^{n}\tau_i P(0,T_{i-1})\left[E^{i-1}\left(S_{i-1,i-1+c}(T_{i-1})\right)-K\right]$$

$$2: \quad \rightarrow=\sum_{i=1}^{n}\tau_i\left(P(0,T_n)\,E^{n}\left(\frac{S_{i-1,i-1+c}(T_{i-1})}{P(T_{i-1},T_n)}\right)\right.$$

$$\left.-KP(0,T_{i-1})\right).$$

CMS's

We need only compute either

$$E^{i-1}\left[S_{i-1,i-1+c}(T_{i-1})\right] \quad \text{or} \quad E^{n}[S_{i-1,i-1+c}(T_{i-1})/P(T_{i-1},T_n)]$$

At first sight, one might think to discretize the dynamics of the forward swap rate in the swap model under the relevant forward measure, and compute the required expectation through a Monte Carlo simulation. However, notice that forward rates appear in the drift of such equation, so that we are forced to evolve forward rates anyway. As a consequence, we can build forward swap rates as functions of the forward LIBOR rates obtained by the Monte Carlo simulated dynamics of the LIBOR model. Find the swap rate $S_{i-1,i-1+c}(T_{i-1})$ from the T_{i-1} values of the (Monte Carlo generated) spanning forward rates

$$F_i(T_{i-1}), F_{i+1}(T_{i-1}), \ldots, F_{i-1+c}(T_{i-1}).$$

Analogously to earlier cases, such forward rates can be generated according to the usual discretized (Milstein) dynamics based on Gaussian shocks and under the unique measure Q^n for example.

CMS's

Alternatively, resort to $S_{\alpha,\beta}(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) F_i(T_\alpha)$ and compute

$$E^\alpha S_{\alpha,\beta}(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) E^\alpha F_i(T_\alpha)$$

$$\approx \sum_{i=\alpha+1}^{\beta} w_i(0) e^{\int_0^{T_\alpha} \bar{\mu}_{\alpha,i}(t)dt} F_i(0)$$

$$E^\alpha S_{\alpha,\beta}(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) E^\alpha F_i(T_\alpha)$$

$$\approx \sum_{i=\alpha+1}^{\beta} w_i(0) e^{\int_0^{T_\alpha} \bar{\mu}_{\alpha,i}(t)dt} F_i(0)$$

We have frozen again the drift in the F_i's dynamics of the F's under Q^α. This can be compared with classical market convexity adjustments. The two methods give similar results when volatilities are not too high. Notation for $\bar{\mu}$ was given at the beginning of this unit.

The method is general and can be used whenever swap rates or forward rates are paid at times that are not "natural" in swaps and similar contracts. A dynamics can be obtained by the freezing procedures outlined above.

That's all Folks!!!

For questions, further references, etc

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For the exam, it is highly recommended to have a look at the exercises done during the course.

For further developments or to expand on the course contents the second edition of "Interest Rate Models: Theory and Practice, With Smile, Inflation and Credit" is a good reference.

http://www.damianobrigo.it/book.html

Thanks for your attention and all the best!!