LECTURE 3

- Calibration: The data.
- Calibration: Instantaneous correlations as inputs or outputs?
- Instantaneous correlations: Calibration Inputs? Historical estimation of correlation matrices and Pivot forms
- Instantaneous correlations: Calibration Outputs? Calibration examples to Caps and Swaptions with diagnostics;
- Instantaneous correlations: Calibration Inputs? Swaptions exact analytical cascade calibration: Practical examples of calibration and diagnostics.
- Advanced Cascade Calibration: impact of the chosen parameterization for instantaneous correlation and of the interpolation technique for missing/illiquid input data

Calibration to swaptions prices

Swaption calibration: Find σ and ρ in LFM such that the LFM reproduces market swaption vols (the first column is T_α and the first row is the underlying swap length $T_\beta - T_\alpha$)

$v^{\text{MKT}}_{\alpha,\beta}$	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	16.4	15.8	14.6	13.8	13.3	12.9	12.6	12.3	12.0	11.7
2y	17.7	15.6	14.1	13.1	12.7	12.4	12.2	11.9	11.7	11.4
3y	17.6	15.5	13.9	12.7	12.3	12.1	11.9	11.7	11.5	11.3
4y	16.9	14.6	12.9	11.9	11.6	11.4	11.3	11.1	11.0	10.8
5y	15.8	13.9	12.4	11.5	11.1	10.9	10.8	10.7	10.5	10.4
7y	14.5	12.9	11.6	10.8	10.4	10.3	10.1	9.9	9.8	9.6
10y	13.5	11.5	10.4	9.8	9.4	9.3	9.1	8.8	8.6	8.4

Table 2: Black vols of EURO ATM swaptions May 16, 2000

Table (brokers) not updated uniformly. Some entries may refer to older market situations.

"Temporal misalignment/Stale data"

Calibrated parameters σ or ρ might reflect this by weird configurations. If so:

Trust the model $\Rightarrow$ detect misalignments

Trust the data $\Rightarrow$ need a better parameterization.

Instantaneous Correlations: Inputs or Outputs?

Swaptions: Fit "Market prices" to "model prices(σ, ρ)".

Should we infer ρ itself from swaption market quotes or should we estimate ρ exogenously and impose it, leaving the calibration only to σ? Are the parameters in ρ inputs or outputs to the calibration?

Inputs? We might consider a time series of past interest-rate curves data, which are observed under the real world probability measure. This would allow us, through interpolation, to obtain a corresponding time series for the particular forward LIBOR rates being modelled in our LIBOR model. These series would be observed under the objective or real-world measure. Thanks to the Girsanov theorem this is not a problem, since instantaneous correlations, considered as instantaneous covariations between driving Brownian motions in forward rate dynamics, do not depend on the probability measure. Then, by using historical estimation, we obtain an historical estimate of the instantaneous correlation matrix. This ρ, or a stylized version of it, can be considered as a given ρ for our LIBOR model, and the remaining free parameters σ are to be used to calibrate market derivatives data. In this case calibration will consist in finding the σ's such that the model (caps and) swaptions prices match the corresponding market prices. In this "matching" procedure (often an optimization) ρ is fixed from the start to the found historical estimate and we play on the volatility parameters σ to achieve our matching.

Instantaneous Correlations: Inputs or Outputs?

Outputs? This second possibility considers instantaneous correlations as fitting parameters. The model swaptions prices are functions of ρ^B, and possibly of some remaining instantaneous volatility parameters, that are forced to match as much as possible the corresponding market swaptions prices, so that the parameters values implied by the market, $\rho^B = \rho^B_{\text{MKT}}$, are found. In the two-factor angles case for example, one obtains the values of $\theta_1, \ldots, \theta_M$ (and of the volatility parameters not determined by the calibration to caps) that are implied by the market.

INPUTS? OUTPUTS? Which of the two methods is preferable? We will consider again this question later on. Now we try and address the issue of determining a decent historical ρ in case we are to decide later for the "inputs" approach.

Inst Corrs as Inputs: The historical matrix

Since European swaptions turn out to be relatively insensitive to instantaneous (rather than terminal) correlation details (e.g. Jäckel and Rebonato (2000)), we may impose a good exogenous instantaneous correlation matrix and subsequently play on volatilities to calibrate swaptions.

Smoothing the rough historically estimated matrix through a parsimonious "pivot" form enjoying desirable properties may guarantee a smooth and regular behaviour of terminal correlations, and slightly more regular σ's when calibrating. This also avoids problems related to outliers, non-synchronous data and discontinuities in correlation surfaces. These and further problems are recalled by Rebonato e Jäckel (1999), that consequently propose to fit a parametric form onto the estimate.

Secondly, the chosen parametric forms may enjoy particularly interesting properties typical of forward rates correlations.

Thirdly, "pivot" forms depend on a low number of parameters, so that we can more easily control the main features of the matrix, detecting those that provoke undesirable anomalous outputs so as to avoid them. Incorporating personal views or recent changes in the market is also easier with pivot forms.

Inst Corrs as Inputs: The historical matrix

"Reduced rank pivot historical correlation matrix":

1. A market historical correlation matrix is estimated;
2. The parameters of a parsimonious form are determined by keeping the historical estimate as a reference;
3. An angles form of the desired rank is fitted to the resulting parsimonious matrix;

Historical Estimation: In estimating correlations, we take into account the particular nature of forward rates in the LMM, characterized by a fixed maturity, contrary to market quotations, where a fixed *time-to-maturity* is usually considered as time passes. We observe from the market, at different times t

$$P(t, t+Z), P(t+1, t+1+Z), \ldots, P(t+n, t+n+Z),$$

where Z is ranging in a standard set of time-to-maturities. We need insead

$$P(t, T), P(t+1, T), \ldots, P(t+n, T),$$

for the maturities T included in the tenor structure of the chosen LMM. Accordingly, a log-interpolation between discount factors has been carried out and only one year of data has been used, since the first forward rate in the family expires in one year from the starting date. These data span from February 1, 2001 to February 1, 2002.

Inst Corrs as Inputs: The historical matrix

From these daily quotations of notional zero-coupon bonds, whose maturities range from one to twenty years from today, we extracted daily log-returns of the annual forward rates involved in the model. Starting from the following usual gaussian approximation

$$\left[\ln\left(\frac{F_1(t+\Delta t)}{F_1(t)}\right), \ldots, \ln\left(\frac{F_{19}(t+\Delta t)}{F_{19}(t)}\right)\right] \sim MN(\mu, V),$$

where $\Delta t = 1$ day, our estimations of the parameters are based on sample mean and covariance for gaussian variables, and are

$$\hat{\mu}_i = \frac{1}{m}\sum_{k=0}^{m-1} \ln\left(\frac{F_i(t_{k+1})}{F_i(t_k)}\right),$$

$$\hat{V}_{i,j} = \frac{1}{m}\sum_{k=0}^{m-1}\left[\left(\ln\left(\frac{F_i(t_{k+1})}{F_i(t_k)}\right) - \hat{\mu}_i\right)\left(\ln\left(\frac{F_j(t_{k+1})}{F_j(t_k)}\right) - \hat{\mu}_j\right)\right],$$

where m is the number of observed log-returns for each rate, so that our estimation of the general correlation element $\rho_{i,j}$ is

$$\hat{\rho}_{i,j} = \frac{\hat{V}_{i,j}}{\sqrt{\hat{V}_{i,i}}\sqrt{\hat{V}_{j,j}}}.$$

Inst Corrs as Inputs: The historical matrix

Resulting matrix:

	1	2	3	4	5	6	7	8	9	10
1	1.000	0.823	0.693	0.652	0.584	0.467	0.290	0.235	0.434	0.473
2	0.823	1.000	0.798	0.730	0.682	0.546	0.447	0.398	0.529	0.566
3	0.693	0.798	1.000	0.764	0.722	0.629	0.472	0.557	0.671	0.610
4	0.652	0.730	0.764	1.000	0.777	0.674	0.577	0.561	0.681	0.701
5	0.584	0.682	0.722	0.777	1.000	0.842	0.661	0.667	0.711	0.734
6	0.467	0.546	0.629	0.674	0.842	1.000	0.774	0.682	0.729	0.688
7	0.290	0.447	0.472	0.577	0.661	0.774	1.000	0.718	0.709	0.647
8	0.235	0.398	0.557	0.561	0.667	0.682	0.718	1.000	0.735	0.659
9	0.434	0.529	0.671	0.681	0.711	0.729	0.709	0.735	1.000	0.748
10	0.473	0.566	0.610	0.701	0.734	0.688	0.647	0.659	0.748	1.000
11	0.331	0.418	0.484	0.562	0.696	0.770	0.648	0.639	0.591	0.632
12	0.432	0.453	0.519	0.593	0.669	0.694	0.619	0.561	0.665	0.675
13	0.288	0.476	0.483	0.581	0.640	0.659	0.714	0.610	0.688	0.704
14	0.230	0.343	0.542	0.498	0.590	0.634	0.619	0.720	0.693	0.634
15	0.259	0.346	0.462	0.499	0.581	0.615	0.628	0.588	0.690	0.636
16	0.206	0.321	0.422	0.478	0.649	0.677	0.663	0.645	0.634	0.651
17	0.227	0.323	0.450	0.488	0.653	0.702	0.638	0.642	0.644	0.625
18	0.293	0.312	0.420	0.439	0.534	0.569	0.524	0.492	0.518	0.524
19	0.245	0.322	0.352	0.354	0.422	0.447	0.375	0.459	0.402	0.399

Inst Corrs as Inputs: The historical matrix

	11	12	13	14	15	16	17	18	19
1	0.331	0.432	0.288	0.230	0.259	0.206	0.227	0.293	0.245
2	0.418	0.453	0.476	0.343	0.346	0.321	0.323	0.312	0.322
3	0.484	0.519	0.483	0.542	0.462	0.422	0.450	0.420	0.352
4	0.562	0.593	0.581	0.498	0.499	0.478	0.488	0.439	0.354
5	0.696	0.669	0.640	0.590	0.581	0.649	0.653	0.534	0.422
6	0.770	0.694	0.659	0.634	0.615	0.677	0.702	0.569	0.447
7	0.648	0.619	0.714	0.619	0.628	0.663	0.638	0.524	0.375
8	0.639	0.561	0.610	0.720	0.588	0.645	0.642	0.492	0.459
9	0.591	0.665	0.688	0.693	0.690	0.634	0.644	0.518	0.402
10	0.632	0.675	0.704	0.634	0.636	0.651	0.625	0.524	0.399
11	1.000	0.832	0.722	0.642	0.581	0.679	0.727	0.566	0.448
12	0.832	1.000	0.819	0.687	0.675	0.704	0.686	0.654	0.426
13	0.722	0.819	1.000	0.785	0.776	0.785	0.715	0.594	0.425
14	0.642	0.687	0.785	1.000	0.820	0.830	0.788	0.599	0.453
15	0.581	0.675	0.776	0.820	1.000	0.901	0.796	0.501	0.222
16	0.679	0.704	0.785	0.830	0.901	1.000	0.939	0.707	0.464
17	0.727	0.686	0.715	0.788	0.796	0.939	1.000	0.818	0.657
18	0.566	0.654	0.594	0.599	0.501	0.707	0.818	1.000	0.836
19	0.448	0.426	0.425	0.453	0.222	0.464	0.657	0.836	1.000

Inst Corrs as Inputs: The historical matrix

Examining the matrix, we see a pronounced and approximately monotonic decorrelation along the columns, when moving away from the diagonal. We see also a relevant initial steepness of the decorrelation pattern. The upward trend along the sub-diagonals is not remarkable. That might be due to the smaller extent of such a phenomenon, more likely to be hidden by noise or differences in liquidity amongst longer rates. Not very different features are visible also in the previous similar estimate showed in Brace, Gatarek and Musiela (1997).

We did some tests on the stability of the estimates, finding out that the values remain rather constant even if we change the sample size or its time positioning.

Inst Corrs as Inputs: The historical matrix

Principal component analysis reveals that 7 factors are required to explain 90% of the overall variability.

1	11,6992	61,575%	61,575%
2	2,1478	11,304%	72,879%
3	1,1803	6,212%	79,091%
4	0,7166	3,772%	82,863%
5	0,6413	3,375%	86,238%
6	0,4273	2,249%	88,487%
7	0,386	2,032%	90,519%
8	0,3389	1,784%	92,303%
9	0,2805	1,476%	93,779%
10	0,2542	1,338%	95,117%
11	0,1995	1,050%	96,167%
12	0,1692	0,891%	97,057%
13	0,1611	0,848%	97,905%
14	0,1503	0,791%	98,696%
15	0,0877	0,462%	99,158%
16	0,0601	0,316%	99,474%
17	0,0515	0,271%	99,745%
18	0,0333	0,175%	99,921%
19	0,0151	0,079%	100,000%

Table 3: Eigenvalues of the historical correlation matrix in decreasing order with single and cumulated percentage variance explained by them

Now we have a realistic correlation benchmark, consistent with market tendencies, to be used as a blueprint for determining the values of the parameters in the presented correlation forms. The methods to do so are tackled in the following.

Inst Corrs as Inputs: Pivot matrices

Here we concentrate on the full rank parameterizations seen earlier (S&C3, Classical Exponential, Rebonato exponential). The classic methodology is fitting the chosen parametric form to the historically estimated matrix by minimizing some loss function of the difference between the two matrices.

Morini (2002) proposes instead to invert directly the functional structure of the parametric forms. Parameters are expressed as functions of key elements of the target historical matrix, so that such elements will be exactly reproduced. We dub such key elements "pivot points" of the historical matrix, and the resulting parametric matrices "pivot matrices". The Pivot approach:

1. does not need any optimization routine;
2. If the pivot points are chosen appropriately, it generates a matrix with the same typical monotonicity and positivity properties as the original one.
3. parameters have a clear, intuitive meaning, since they are expressed in terms of correlation entries considered to be particularly significant. This allows us to easily alter and deform the matrix playing with the parameters in a controlled way, as might be needed in the market practice.
4. It keeps out the negative effects of irregularities and clear outliers typical of historical estimations.
5. In our examples the fitting error with the Pivot method is not so far from the error in a complete, optimal fitting.

Inst Corrs as Inputs: Pivot matrices

Pivot points must be chosen carefully. We will start by considering three-parameters structures. We consider the entries $\rho_{1,2}$, $\rho_{1,M}$ and $\rho_{M-1,M}$. Such elements embed basic monotonicity information of the historical correlation matrix.

Morini (2002) computes, starting with Rebonato's exponential form,

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp[-|i-j|(\beta - \alpha(\max(i,j)-1))], \ \ \beta \geq 0.$$

the equations

$$\left(\frac{\rho_{1,M} - \rho_\infty}{1-\rho_\infty}\right) = \left(\frac{\rho_{M-1,M} - \rho_\infty}{1-\rho_\infty}\right)^{(M-1)},$$

for ρ_∞, and

$$\alpha = \frac{\ln\left(\frac{\rho_{1,2} - \rho_\infty}{\rho_{M-1,M} - \rho_\infty}\right)}{2-M}, \qquad \beta = \alpha - \ln\left(\frac{\rho_{1,2} - \rho_\infty}{1-\rho_\infty}\right).$$

The results are

$$\rho_\infty = 0.23551, \quad \alpha = 0.00126, \quad \beta = 0.26388.$$

Inst Corrs as Inputs: Pivot matrices

Let us now move on to form SC3,

$$\rho_{i,j} = \exp\left[-|i-j|\left(\beta - \frac{\alpha_2}{6M-18}\left(i^2+j^2+ij-6i-6j-3M^2+15M-7\right)\right.\right.$$
$$\left.\left.+\frac{\alpha_1}{6M-18}\left(i^2+j^2+ij-3Mi-3Mj+3i+3j+3M^2-6M+2\right)\right)\right]. \tag{8}$$

Morini computes

$$\beta = -\ln\left(\rho_{M-1,M}\right).$$

and

$$\alpha_1 = \frac{6\ln\rho_{1,M}}{(M-1)(M-2)} - \frac{2\ln\rho_{M-1,M}}{(M-2)} - \frac{4\ln\rho_{1,2}}{(M-2)},$$
$$\alpha_2 = -\frac{6\ln\rho_{1,M}}{(M-1)(M-2)} + \frac{4\ln\rho_{M-1,M}}{(M-2)} + \frac{2\ln\rho_{1,2}}{(M-2)},$$

leading to

$$\alpha_1 = 0.03923, \quad \alpha_2 = -0.03743, \quad \beta = 0.17897.$$

Inst Corrs as Inputs: Pivot matrices

Consider also the pivot version of S&C2:

$$\rho_{i,j} = \exp\Bigg[\quad - \quad \frac{|i-j|}{M-1}\Bigg(- \ln \rho_\infty + \quad \eta \frac{i^2 + j^2 + ij - 3Mi - 3Mj + 3i + 3j + 2M^2 - M - 4}{(M-2)(M-3)} \Bigg)\Bigg].$$

Use as pivot points $\rho_{1,M}$ and $\rho_{1,2}$. $\rho_{1,2}$ is selected for reasons that will be clear later on. We have

$$\rho_\infty = \rho_{1,M}, \quad \eta = \frac{(-\ln \rho_{1,2})\,(M-1) + \ln \rho_\infty}{2},$$

and obtain $\rho_\infty = 0.24545, \quad \eta = 1.04617.$

We compared the two three-parameters pivot forms with respect to the goodness of fit (to the historical matrix). S&C3 pivot is superior when we take as loss function the simple average squared difference (denoted by MSE), whilst Rebonato pivot is better if considering the average squared *relative* difference with respect to the estimated matrix (denoted by MSE%). This is shown in the following table.

	MSE	MSE%	$\sqrt{\text{MSE}}$	$\sqrt{\text{MSE\%}}$
Reb. 3 pivot	0.030121	0.09542	0.173554	0.30890
S&C3 pivot	0.024127	0.10277	0.155327	0.32058

Inst Corrs as Inputs: Pivot matrices

Some reasons for considering Rebonato pivot form preferable in this context arise from the graphical observation of the behaviour of these matrices. As visible in the first figure below, showing the plot of the first columns, such matrix seems a better approximation of the estimated tendency, whereas S&C3 pivot tends to keep higher than the historical matrix. Moreover, in matching the estimated values selected, the parameter α_2 in S&C3 has turned out to be negative. This has led to a non-monotonic trend for sub-diagonals, see in fact the humped shape for the first sub-diagonal.

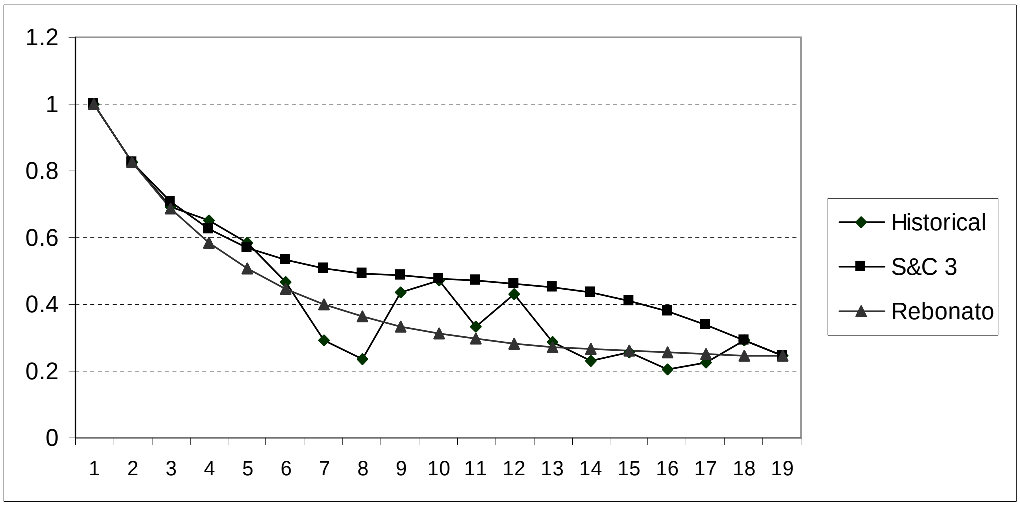


Figure 4: First columns of the historical and fitted "pivot" matrices

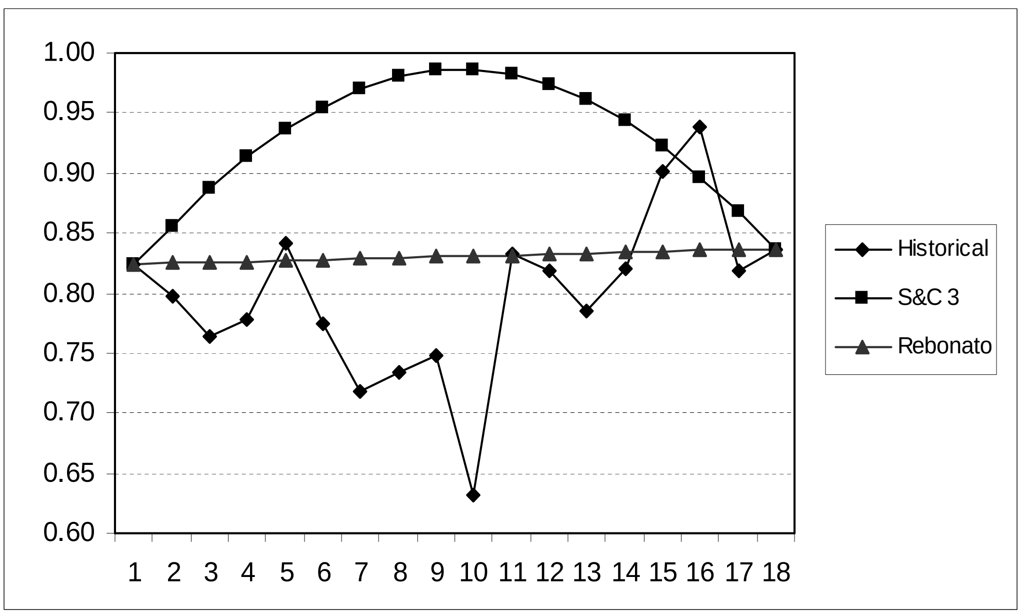


Figure 5: Corresponding sub-diagonals

Inst Corrs as Inputs: Pivot matrices

A similar problem is hinted at also by Schoenmakers and Coffey (2000). In their constrained tests α_2 tends to assume always the minimum value allowed, namely zero. They propose the form S&C2. Our results with S&C2 pivot suggest that this very faint increasing tendency along sub-diagonals, joined with the level of decorrelation along the columns seen in the historical estimate, represent a configuration very hard to replicate with S&C parameterizations.
Indeed, by building a pivot S&C2 keeping out information upon the sub-diagonal behaviour, one gets a matrix spontaneously featuring a *strong* increase along such sub-diagonals. On the other hand, including information on this estimated behaviour, a far larger decorrelation is implied than in the historically estimated matrix. More elements and details on such tests are given in Morini (2002).

No such problem has emerged for Rebonato's 3 parameters pivot form, that seems to allow for an easier separation of the tendency along sub-diagonal from the one along the columns. Moreover, notice that Rebonato pivot form, with our data, turns out to be positive definite, so that its main theoretical limitation does not represent a problem in practice.

Our preferred choice is Rebonato-exponential 3-parameters.

Inst Corrs as Inputs: Pivot matrices

Now we have still to check the divergence between pivot matrices and matrices optimally fitted to the entire target matrix.

We compare the pivot version of Rebonato's parameterization with two optimal specifications of the same form obtained by minimizing the aforementioned loss functions. In the following table we present for each optimal form the square root of the corresponding error, besides the value obtained, for the same measure, when considering the pivot form.

	$\sqrt{\text{MSE}}$	$\sqrt{\text{MSE\%}}$
Fitted vs Historical	0.108434	0.25949
Pivot vs Historical	0.173554	0.30890

Differences are relatively small. First columns are plotted below.

Inst Corrs as Inputs: Pivot matrices

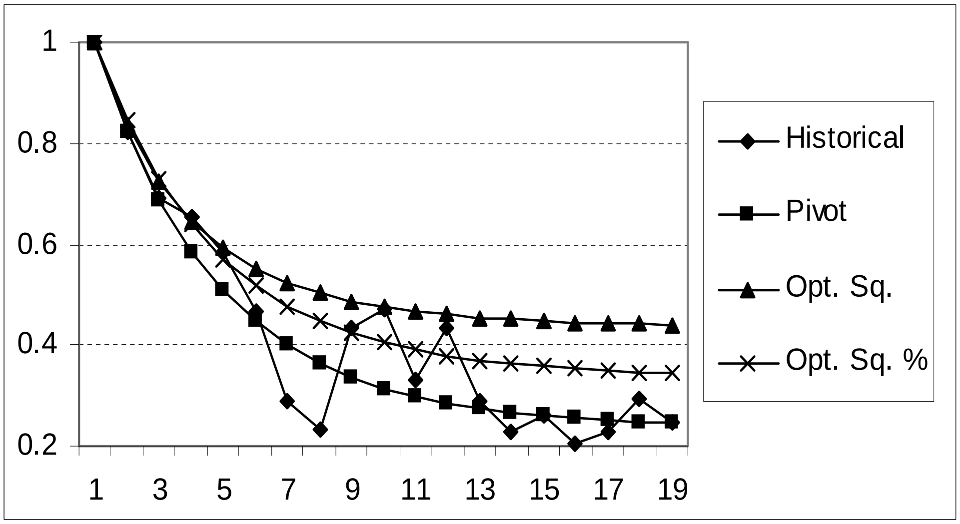


Figure 6: First columns of correlation matrices

Conclusion: The pivot approach can be helpful when trying to describe the essential stylized feature of the historical correlation matrix. The related matrix, or a reduced rank version of it, can be considered as a reasonable exogenous correlation matrix to be used as input for calibration to (caps and) swaptions.

Inst Corrs as Outputs: Joint calibration to caps and swaptions

We start with ρ as calibration outputs.

CALIBRATION: Need to find $\sigma(t)$ and ρ such that the market prices of caps and swaptions are recovered by LFM(σ, ρ).

caplet-volat-LFM(σ)= market-caplet-volat (Almost automatic).

swaptions-LFM(σ, ρ)= market-swaptions.

Caplets: Algebraic formula; Immediate calibration, almost automatic.

Swaptions: In principle Monte Carlo pricing. But MC pricing at each optimization step is too computationally intensive.

Use Rebonato's approximation and at each optimization step evaluate swaptions analytically with the LFM model.

ρ as outputs. Joint calibration: Market cases

SPC vols, $\sigma_k(t) = \sigma_{k,\beta(t)} := \Phi_k \psi_{k-(\beta(t)-1)}$.

ρ rank-2 with angles $-\pi/2 < \theta_i - \theta_{i-1} < \pi/2$

Data below as of May 16, 2000, $F(0;0,1y) = 0.0469$, plus swaptions matrix as in the earlier slide.

Index	initial $F0$	v_{caplet}
1	0.050114	0.180253
2	0.055973	0.191478
3	0.058387	0.186154
4	0.060027	0.177294
5	0.061315	0.167887
6	0.062779	0.158123
7	0.062747	0.152688
8	0.062926	0.148709
9	0.062286	0.144703
10	0.063009	0.141259
11	0.063554	0.137982
12	0.064257	0.134708
13	0.064784	0.131428
14	0.065312	0.128148
15	0.063976	0.127100
16	0.062997	0.126822
17	0.061840	0.126539
18	0.060682	0.126257
19	0.059360	0.125970

Index	ψ	Φ	θ
1	2.5114	0.0718	1.7864
2	1.5530	0.0917	2.0767
3	1.2238	0.1009	1.5122
4	1.0413	0.1055	1.6088
5	0.9597	0.1074	2.3713
6	1.1523	0.1052	1.6031
7	1.2030	0.1043	1.1241
8	0.9516	0.1055	1.8323
9	1.3539	0.1031	2.3955
10	1.1912	0.1021	2.5439
11	0	0.1046	1.6118
12	3.3778	0.0844	1.3172
13	0	0.0857	1.2225
14	1.2223	0.0847	1.0995
15	0	0.0869	1.2602
16	0	0.0896	1.0905
17	0	0.0921	0.8006
18	0.1156	0.0946	0.8739
19	0.5753	0.0965	1.7096

ρ as outputs. Joint calibration: Market cases (cont'd)

Quality of calibration: Caplets are fitted exactly, whereas we calibrated the whole swaptions volatility matrix except for the first column.

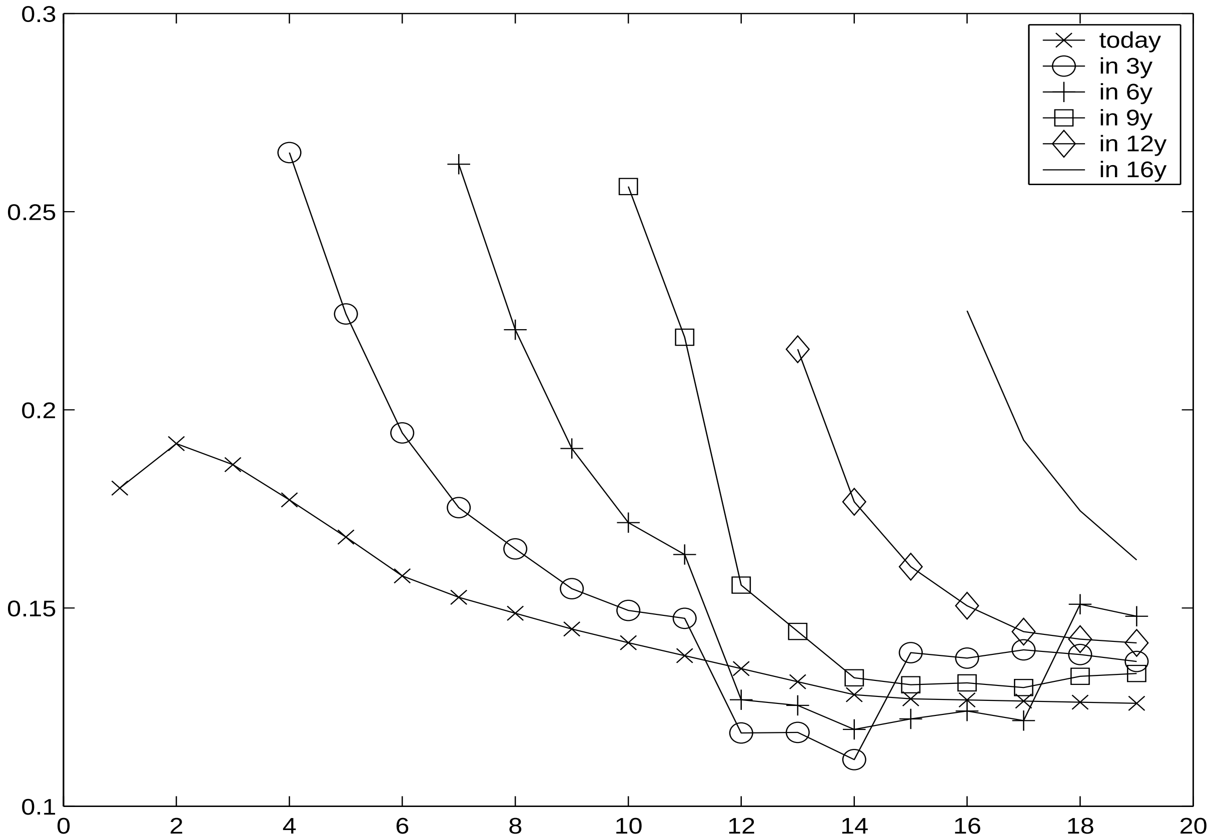
Matrix: 100(Mkt swaptions vol - LFM swaption vol)/Mkt swaptions vol:

	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	-0.71	0.90	1.67	4.93	3.00	3.25	2.81	0.83	0.11
2y	-2.43	-3.48	-1.54	-0.70	0.70	0.01	-0.22	-0.45	0.49
3y	-3.84	1.28	-2.44	-0.69	-1.18	0.21	1.51	1.57	-0.01
4y	1.87	-2.52	-2.65	-3.34	-2.17	-0.44	-0.11	-0.63	-0.38
5y	1.80	4.15	-1.40	-1.89	-1.74	-0.79	-0.34	-0.07	1.28
7y	-0.33	2.27	1.47	-0.97	-0.77	-0.65	-0.57	-0.15	0.19
10y	-0.02	0.61	0.45	-0.31	0.02	-0.03	0.01	0.23	-0.30

Calibr error OK for 19 caplets and 63 swaptions, but... calibrated θ's imply erratic, oscillating (+/-) ρ's and 10y terminal correlations:

	10y	11y	12y	13y	14y	15y	16y	17y	18y	19
10y	1.00	0.56	0.27	0.19	0.09	0.21	0.08	-0.10	-0.06	0.37
11y	0.56	1.00	0.61	0.75	0.67	0.68	0.64	0.44	0.42	0.50
12y	0.27	0.61	1.00	0.42	0.71	0.53	0.48	0.43	0.40	0.42
13y	0.19	0.75	0.42	1.00	0.36	0.71	0.50	0.41	0.43	0.34
14y	0.09	0.67	0.71	0.36	1.00	0.32	0.67	0.43	0.40	0.36
15y	0.21	0.68	0.53	0.71	0.32	1.00	0.28	0.59	0.39	0.33
16y	0.08	0.64	0.48	0.50	0.67	0.28	1.00	0.22	0.62	0.30
17y	-0.10	0.44	0.43	0.41	0.43	0.59	0.22	1.00	0.17	0.36
18y	-0.06	0.42	0.40	0.43	0.40	0.39	0.62	0.17	1.00	0.07
19y	0.37	0.50	0.42	0.34	0.36	0.33	0.30	0.36	0.07	1.00

Joint calibration: Market cases (cont'd)



Evolution of the term structure of caplet volatilities
Loses the "humped shape" after a short time.
Becomes somehow "noisy"
+ previous results on fitted correlation: future market structures implied by the fitted model are not regular under SPC.

Joint calibration: Market cases (cont'd)

Tried other calibrations with SPC σ's

Tried: More stringent constraints on the θ

Fixed θ both to typical and atypical values, leaving the calibration only to the vol parameters

Fixed θ so as to have all $\rho = 1$.

Summary: To have good calibration to swaptions need to keep the angles unconstrained and allow for partly oscillating ρ's.

If we force "smooth/monotonic" ρ's and leave calibr to vols, results are essentially the same as in the case of a one-factor LFM with $\rho = 1$.

Maybe inst correlations do not have a strong link with European swaptions prices? (Rebonato)

Maybe permanence of "bad results", no matter the particular "smooth" choice of fixed ρ, reflects an impossibility of a low-rank ρ to decorrelate quickly fwd rates in a steep initial pattern? (Rebonato)

3-4 factor ρ's does not seem to help. Increase drastically # factors? But MC... More on this later.

Joint calibration: Market cases (cont'd)

Calibration with the LE parametric σ's.

Same inputs as before

Rank-2 ρ with $-\pi/3 < \theta_i - \theta_{i-1} < \pi/3, \quad 0 < \theta_i < \pi$

Constraint "$1 - 0.1 \leq \Phi_i(a, b, c, d) \leq 1 + 0.1$"

Calibrated parameters and calibration error (caps exact):

$$a = 0.29342753, \quad b = 1.25080230, \quad c = 0.13145869, \quad d = 0.00,$$

$$\theta_{1\div 7} = [1.75411\ 0.57781\ 1.68501\ 0.58176\ 1.53824\ 2.43632\ 0.88011],$$

$$\theta_{8\div 12} = [1.89645\ 0.48605\ 1.28020\ 2.44031\ 0.94480],$$

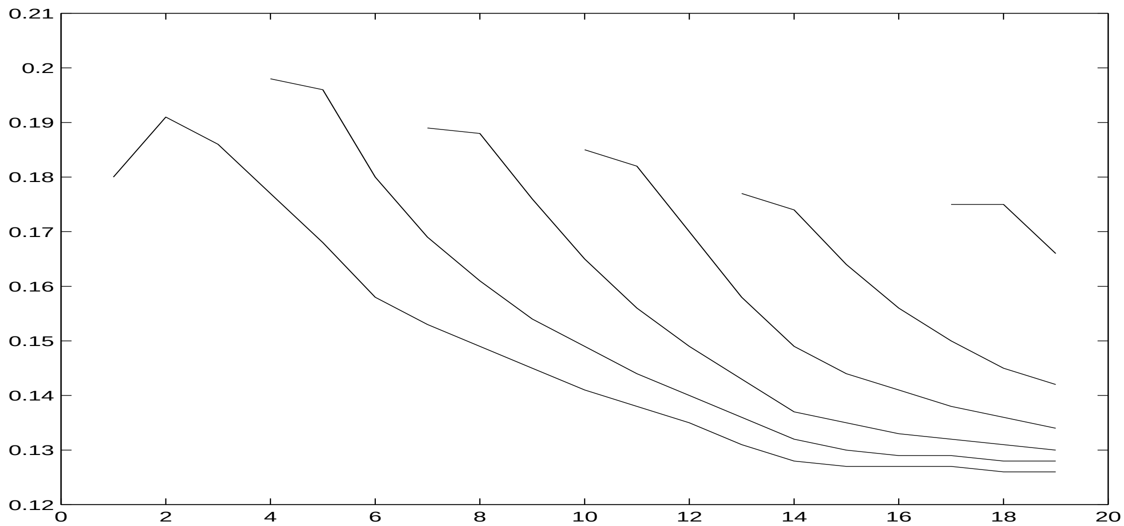
$$\theta_{13\div 19} = [1.34053\ 2.91133\ 1.99622\ 0.70042\ 0\ 0.81518\ 2.38376].$$

Calibration error:

	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	2.28	-3.74	-3.19	-4.68	2.46	1.50	0.72	1.33	-1.42
2y	-1.23	-7.67	-9.97	2.10	0.49	1.33	1.56	-0.44	1.88
3y	2.23	-6.20	-1.30	-1.32	-1.43	1.86	-0.19	2.42	1.17
4y	-2.59	9.02	1.70	0.79	3.22	1.19	4.85	3.75	1.21
5y	-3.26	-0.28	-8.16	-0.81	-3.56	-0.23	-0.08	-2.63	2.62
7y	0.10	-2.59	-10.85	-2.00	-3.67	-6.84	2.15	1.19	0.00
10y	0.29	-3.44	-11.83	-1.31	-4.69	-2.60	4.07	1.11	0.00

Inst correlations are again oscillating and non-monotonic. Terminal correlations share part of this negative behaviour.

Joint calibration: Market cases (cont'd)



Evolution of term structure of vols looks better

Many more experiments with rank-three correlations, less or more stringent constraints on the angles and on the Φ's.

Fitting to the whole swaption matrix can be improved, but at the cost of an erratic behaviour of both correlations and of the evolution of the term structure of volatilities in time.

3-factor choice does not seem to help that much, as before. LE σ's allow for an easier control of the evolution of the term structure of vols, but produce more erratic ρ's: most of the "noise" in the swaption data ends up in the angles (we have only 4 vol parameters a, b, c, d for fitting swaptions)

Cascade Calibration with GPC vols

ρ's as **inputs** to the calibration (e.g. historical estimation)

$$(v^{\mathsf{LFM}}_{\alpha,\beta})^2 \approx \frac{1}{T_\alpha}\sum_{i,j=\alpha+1}^{\beta}\frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2}\int_0^{T_\alpha}\sigma_i(t)\sigma_j(t)\,dt,$$

$$T_\alpha\ S_{\alpha,\beta}(0)^2\ v^2_{\alpha,\beta} =$$

$$= \sum_{i,j=\alpha+1}^{\beta-1} w_i w_j F_i F_j \rho_{i,j} \sum_{h=0}^{\alpha}(T_h - T_{h-1})\ \sigma_{i,h+1}\ \sigma_{j,h+1}$$

$$+ \ 2\sum_{j=\alpha+1}^{\beta-1} w_\beta w_j F_\beta F_j \rho_{\beta,j} \sum_{h=0}^{\alpha-1}(T_h - T_{h-1})\ \sigma_{\beta,h+1}\ \sigma_{j,h+1}$$

$$+ \ 2\sum_{j=\alpha+1}^{\beta-1} w_\beta w_j F_\beta F_j \rho_{\beta,j}\ (T_\alpha - T_{\alpha-1}) \boxed{\ \sigma_{\beta,\alpha+1}\ }\ \sigma_{j,\alpha+1}$$

$$+ \ w^2_\beta F^2_\beta \sum_{h=0}^{\alpha-1}(T_h - T_{h-1})\ \sigma^2_{\beta,h+1}$$

$$+ \ w^2_\beta F^2_\beta (T_\alpha - T_{\alpha-1}) \boxed{\ \sigma^2_{\beta,\alpha+1}\ }\ .$$

Solve this 2nd order eq: all quantities known or previously calculated except $\sigma_{\beta,\alpha+1}$, *provided that* the "upper diagonal part" of the input swaption matrix is visited left to right and top down, starting from the upper left corner ($v_{0,1} = \sigma_{1,1}$.)

Cascade Calibr with general PC vols: One to one corresp with swaption vols (cont'd)

Length / Maturity	1y	2y	3y
$T_0 = 1$y	$v_{0,1}$ $\sigma_{1,1}$	$v_{0,2}$ $\sigma_{1,1}, \sigma_{2,1}$	$v_{0,3}$ $\sigma_{1,1}, \sigma_{2,1}, \sigma_{3,1}$
$T_1 = 2$y	$v_{1,2}$ $\sigma_{2,1}, \sigma_{2,2}$	$v_{1,3}$ $\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \sigma_{3,2}$	- -
$T_2 = 3$y	$v_{2,3}$ $\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}$	-	-

Problem: can obtain negative or imaginary σ's.

Possible cause: Illiquidity/stale data on the v's.

Possible remedy: Smooth the input swaption v's matrix with a 17-dimensional parametric form and recalibrate: imaginary and negative vols σ disappear.

Term structure of caplet vols evolves regularly but loses hump

Instantaneous correlations good because chosen exogenously

Terminal correlations positive and monotonically decreasing

This form can help in Vega breakdown analysis (helpful for hedging)

Exact Swaption Cascade Calibration with GPC: Numerical example

Calibrate σ's to the following swaptions matrix (2000)

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	0.180	0.167	0.154	0.145	0.138	0.134	0.130	0.126	0.124	0.122
2y	0.181	0.162	0.145	0.135	0.127	0.123	0.120	0.117	0.115	0.113
3y	0.178	0.155	0.137	0.125	0.117	0.114	0.111	0.108	0.106	0.104
4y	0.167	0.143	0.126	0.115	0.108	0.105	0.103	0.100	0.098	0.096
5y	0.154	0.132	0.118	0.109	0.104	0.104	0.099	0.096	0.094	0.092
6y	0.147	0.127	0.113	0.104	0.098	0.098	0.094	0.092	0.090	0.089
7y	0.140	0.121	0.107	0.098	0.092	0.091	0.089	0.087	0.086	0.085
8y	0.137	0.117	0.103	0.095	0.089	0.088	0.086	0.084	0.083	0.082
9y	0.133	0.114	0.100	0.091	0.086	0.085	0.083	0.082	0.081	0.080
10y	0.130	0.110	0.096	0.088	0.083	0.082	0.080	0.079	0.078	0.077

added vols for 6y,8y and 9y maturities by linear interpolation.

assume nice decreasing positive rank 2 corr given *exogenously*, $\rho_{i,j} = \cos(\theta_i - \theta_j)$, corresponding to the angles

$$\theta_{1\div 9} = [\ 0.0147\ 0.0643\ 0.1032\ 0.1502\ 0.1969\ 0.2239\ 0.2771\ 0.2950\ 0.3630\ ],$$

$$\theta_{10\div 19} = [\ 0.3810\ 0.4217\ 0.4836\ 0.5204\ 0.5418\ 0.5791\ 0.6496\ 0.6679\ 0.7126\ 0.7659\ ].$$

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd)

0.1800	-	-	-	-	-	-	-	-	-
0.1548	0.2039	-	-	-	-	-	-	-	-
0.1285	0.1559	0.2329	-	-	-	-	-	-	-
0.1178	0.1042	0.1656	0.2437	-	-	-	-	-	-
0.1091	0.0988	0.0973	0.1606	0.2483	-	-	-	-	-
0.1131	0.0734	0.0781	0.1009	0.1618	0.2627	-	-	-	-
0.1040	0.0984	0.0502	0.0737	0.1128	0.1633	0.2633	-	-	-
0.0940	0.1052	0.0938	0.0319	0.0864	0.0969	0.1684	0.2731	-	-
0.1065	0.0790	0.0857	0.0822	0.0684	0.0536	0.0921	0.1763	0.2848	-
0.1013	0.0916	0.0579	0.1030	0.1514	- 0.0316	0.0389	0.0845	0.1634	0.2777
0.0916	0.0916	0.0787	0.0431	0.0299	0.2088	- 0.0383	0.0746	0.0948	0.1854
0.0827	0.0827	0.0827	0.0709	0.0488	0.0624	0.1561	- 0.0103	0.0731	0.0911
0.0744	0.0744	0.0744	0.0744	0.0801	0.0576	0.0941	0.1231	- 0.0159	0.0610
0.0704	0.0704	0.0704	0.0704	0.0704	0.1009	0.0507	0.0817	0.1203	- 0.0210
0.0725	0.0725	0.0725	0.0725	0.0725	0.0725	0.1002	0.0432	0.0619	0.1179
0.0753	0.0753	0.0753	0.0753	0.0753	0.0753	0.0753	0.0736	0.0551	0.0329
0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0708	0.0702
0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0680
0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663

Calibration shows negative signs in σ's. "Temporal misalignments" caused by illiquidity in the swaption matrix? In some cases one can also have complex volatilities. To avoid this, smooth the market swaption matrix by fitting

$$\text{vol}(S,T) = \gamma(S) + \left(\frac{\exp(f \cdot \ln(T))}{e \cdot S} + D(S)\right) \cdot \exp(-\beta \cdot \exp(p \cdot \ln(T))),$$

where (S is the maturity, T the tenor)

$$\begin{aligned} \gamma(S) &= c + (\exp(h \cdot \ln(S)) \cdot a + d) \cdot \exp(-b \cdot \exp(m \cdot \ln(S))), \\ D(S) &= (\exp(g \cdot \ln(S)) \cdot q + r) \cdot \exp(-s \cdot \exp(t \cdot \ln(S))) + \delta, \end{aligned}$$

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd)

a	b	c	d	e	f	del	bet
0.000359	1.432288	2.5269	-1.93552	5.751286	0.065589	0.02871	-5.41842
g	h	m	p	q	r	s	t
-0.02129	17.64259	2.043768	-0.06907	-0.09817	-0.87881	2.017844	0.600784

Difference between the market and the smoothed matrices:

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	-0.46	0.49	0.33	0.16	-0.01	0.01	-0.06	-0.18	-0.14	-0.14
2y	-0.39	0.53	0.18	0.03	-0.17	-0.11	-0.05	-0.05	0.01	0.03
3y	0.03	0.64	0.22	-0.13	-0.32	-0.16	-0.10	-0.10	-0.05	-0.03
4y	0.01	0.43	0.05	-0.23	-0.35	-0.21	-0.06	-0.08	-0.04	-0.03
5y	-0.36	0.12	-0.02	-0.15	-0.10	0.31	0.14	0.11	0.14	0.13
6y	-0.31	0.19	-0.02	-0.18	-0.21	0.13	0.09	0.10	0.16	0.20
7y	-0.27	0.25	-0.01	-0.21	-0.32	-0.05	0.05	0.09	0.19	0.27
8y	-0.13	0.27	-0.04	-0.22	-0.32	-0.06	0.02	0.09	0.18	0.25
9y	0.00	0.30	-0.05	-0.24	-0.32	-0.07	0.00	0.10	0.18	0.25
10y	0.15	0.32	-0.07	-0.25	-0.31	-0.08	-0.02	0.09	0.17	0.23

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd)

σ's obtained calibrating the smoothed swaption matrix:

18.46	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
14.09	22.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12.84	13.11	24.71	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12.14	11.17	13.00	25.94	0.00	0.00	0.00	0.00	0.00	0.00
11.64	10.11	10.59	12.54	27.10	0.00	0.00	0.00	0.00	0.00
11.19	9.51	9.44	9.87	12.73	28.06	0.00	0.00	0.00	0.00
10.94	8.88	8.47	8.53	9.82	13.01	28.58	0.00	0.00	0.00
10.59	8.61	7.82	7.57	8.58	10.06	12.92	29.62	0.00	0.00
10.37	8.25	7.53	6.81	7.52	8.61	9.74	13.51	30.20	0.00
10.26	7.73	7.21	6.43	7.14	7.65	8.31	10.45	13.56	30.35
8.89	8.89	7.08	6.31	6.39	7.23	7.38	8.73	10.40	13.41
8.07	8.07	8.07	6.23	6.30	6.82	6.79	7.96	8.63	10.10
7.35	7.35	7.35	7.35	6.27	6.43	6.29	7.38	7.96	8.44
7.01	7.01	7.01	7.01	7.01	6.39	5.85	6.89	6.70	7.46
6.53	6.53	6.53	6.53	6.53	6.53	6.29	5.96	6.92	6.68
6.23	6.23	6.23	6.23	6.23	6.23	6.23	6.97	5.58	6.57
6.06	6.06	6.06	6.06	6.06	6.06	6.06	6.06	6.57	5.77
5.76	5.76	5.76	5.76	5.76	5.76	5.76	5.76	5.76	6.35
5.62	5.62	5.62	5.62	5.62	5.62	5.62	5.62	5.62	5.62

irregularity and illiquidity in the input swaption matrix can cause negative or even imaginary values in the calibrated σ's. However, by smoothing the input data before calibration, usually this undesirable features can be avoided.

Exact Swaption calibration with GPC: Numerical example (cont'd)

The smoothing procedure also improves terminal correlations.

Ten-years terminal correlations for the non-smoothed case:

	10y	11y	12y	13y	14y	15y	16y	17y	18y	19y
10y	1.000	0.677	0.695	0.640	0.544	0.817	0.666	0.762	0.753	0.740
11y	0.677	1.000	0.614	0.617	0.665	0.768	0.696	0.760	0.752	0.740
12y	0.695	0.614	1.000	0.758	0.716	0.938	0.848	0.870	0.862	0.850
13y	0.640	0.617	0.758	1.000	0.740	0.866	0.914	0.894	0.885	0.875
14y	0.544	0.665	0.716	0.740	1.000	0.771	0.919	0.885	0.879	0.868
15y	0.817	0.768	0.938	0.866	0.771	1.000	0.923	0.965	0.960	0.953
16y	0.666	0.696	0.848	0.914	0.919	0.923	1.000	0.983	0.980	0.975
17y	0.762	0.760	0.870	0.894	0.885	0.965	0.983	1.000	0.999	0.995
18y	0.753	0.752	0.862	0.885	0.879	0.960	0.980	0.999	1.000	0.999
19y	0.740	0.740	0.850	0.875	0.868	0.953	0.975	0.995	0.999	1.000

Compare with the corresponding matrix from smoothed data

	10y	11y	12y	13y	14y	15y	16y	17y	18y	19y
10y	1.000	0.939	0.898	0.872	0.851	0.838	0.823	0.809	0.817	0.787
11y	0.939	1.000	0.992	0.980	0.969	0.962	0.947	0.941	0.936	0.915
12y	0.898	0.992	1.000	0.996	0.990	0.986	0.975	0.972	0.966	0.950
13y	0.872	0.980	0.996	1.000	0.997	0.995	0.986	0.984	0.979	0.966
14y	0.851	0.969	0.990	0.997	1.000	0.997	0.992	0.989	0.984	0.973
15y	0.838	0.962	0.986	0.995	0.997	1.000	0.994	0.995	0.990	0.982
16y	0.823	0.947	0.975	0.986	0.992	0.994	1.000	0.997	0.997	0.992
17y	0.809	0.941	0.972	0.984	0.989	0.995	0.997	1.000	0.998	0.995
18y	0.817	0.936	0.966	0.979	0.984	0.990	0.997	0.998	1.000	0.998
19y	0.787	0.915	0.950	0.966	0.973	0.982	0.992	0.995	0.998	1.000

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd)

non-smoothed case is worse: terminal correlations deviate more from monotonicity, roughly corresponding to the portion of instantaneous volatilities that go negative in the calibration. The non-smoothed case shows also a slightly erratic evolution of the term structure of volatilities compared to the smoothed case.

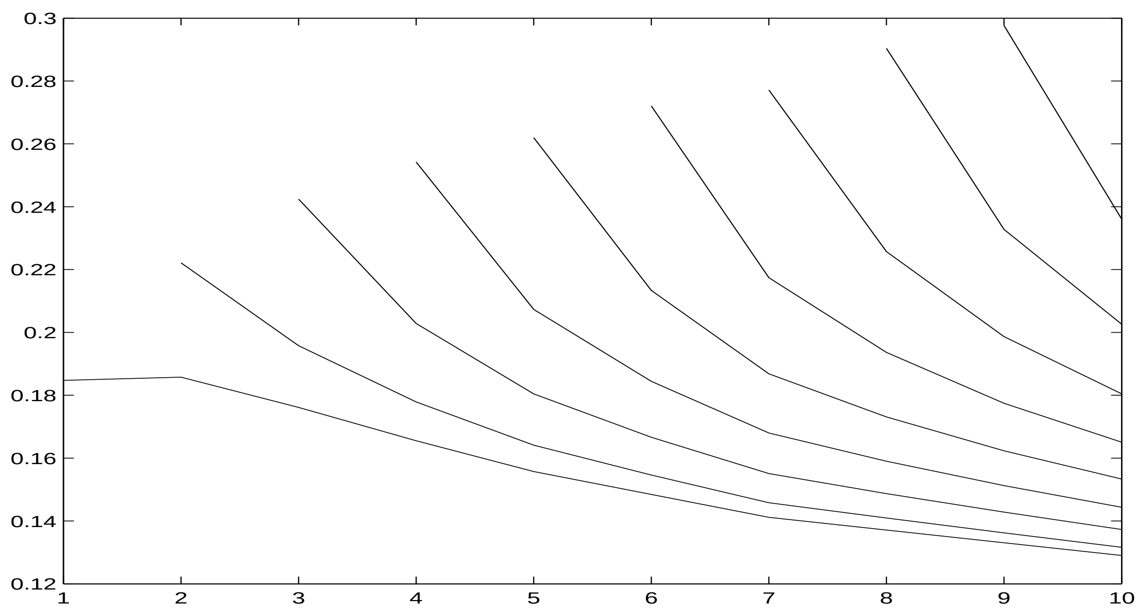


Figure 7: Term structure evolution corresponding to the smoothed volatility swaption data

Calibration: A pause for thought and a First summary

Some desired calibration features:

- A small rank for ρ in view of Monte Carlo
- A small calibration error;
- Positive and decreasing inst. and term. correlations;
- Smooth and stable evolution of the term structure of vols;

Can achieve these targets through a low # of factors?
Try and combine many of the ideas presented here
The one-to-one formulation is perhaps the most promising: Fitting to swaptions is exact; can fit caps by introducing infra-correlations; instantaneous correlation OK by construction; Terminal correlation not spoiled by the fitted σ's; Terms structure evolution smooth but not fully satisfactory qualitatively.
Further investigation in progress (Massimo Morini)
Requirements hardly checkable with general HJM or short-rate models

More mathematically-advanced issues: **Smile calibration**.

Cascade calibration: further developments

The examples and considerations given here are based on more recent market data and have appeared earlier Morini (2002) and in Brigo and Morini (2002).
New Data: input swaption matrix, 1 feb 02.

	1	2	3	4	5	6	7	8	9	10
1	**17.90**	**16.50**	**15.30**	**14.40**	**13.70**	**13.20**	**12.80**	**12.50**	**12.30**	**12.00**
2	**15.40**	**14.20**	**13.60**	**13.00**	**12.60**	**12.20**	**12.00**	**11.70**	**11.50**	11.30
3	**14.30**	**13.30**	**12.70**	**12.20**	**11.90**	**11.70**	**11.50**	**11.30**	11.10	10.90
4	**13.60**	**12.70**	**12.10**	**11.70**	**11.40**	**11.30**	**11.10**	10.90	10.80	10.70
5	**12.90**	**12.10**	**11.70**	**11.30**	**11.10**	**10.90**	10.80	10.60	10.50	10.40
6	**12.50**	**11.80**	**11.40**	**10.95**	**10.75**	*10.60*	*10.50*	*10.40*	*10.35*	*10.25*
7	**12.10**	**11.50**	**11.10**	**10.60**	10.40	10.30	10.20	10.20	10.20	10.10
8	**11.80**	**11.20**	**10.83**	*10.40*	*10.23*	*10.17*	*10.10*	*10.10*	*10.07*	*10.00*
9	**11.50**	**10.90**	*10.57*	*10.20*	*10.07*	*10.03*	*10.00*	*10.00*	*9.93*	*9.90*
10	**11.20**	10.60	10.30	10.00	9.90	9.90	9.90	9.90	9.80	9.80

The annualized forward LIBOR rates from the corresponding zero curve on the same date are

$F(0;0,1)$: 1	0.036712	11	0.058399
$F(0;1,2)$: 2	0.04632	12	0.058458
$\ldots$: 3	0.050171	13	0.058569
4	0.05222	14	0.058339
5	0.054595	15	0.057951
6	0.056231	16	0.057833
7	0.057006	17	0.057555
8	0.057699	18	0.057297
9	0.05691	19	0.056872
10	0.057746	20	0.056738

Cascade Calibration of Rectangular swaption matrices

The rows associated with the swaptions maturities of $6, 8$ and 9 years do not refer to market quotations. Considering that the Cascade Calibration Algorithm (CCA) requires a complete swaption matrix, featuring values for each and every maturity (and length) in the range, they have been obtained as before by a simple linear interpolation between the adjacent market values on the same columns, see also Rebonato and Joshi (2001). We discuss the interpolation effects later.

An important point about the basic CCA given earlier is that *results are, in a sense, independent of the matrix size*, in that the output of the calibration to a sub-matrix will be a subset of the output of a calibration to the original matrix.
This implies also that any swaption matrix V can be seen in principle as a sub-matrix of a larger one, say $\bar{V}$, including V itself in its upper triangular part, so that all entries of V, including those in its lower triangular part, will be recovered by applying the basic CCA algorithm to the upper part of the larger matrix $\bar{V}$. In other words, this "nested consistency" means that, if all needed market values were available, so that we could *always* embed our given market V in a sufficiently large market $\bar{V}$, the basic "upper part" CCA seen earlier might be considered to be general, with no need for any extension.

Cascade Calibration of Rectangular swaption matrices

Of course this is not usually the case, in that in general there is no larger $\bar{V}$ to be exploited.

If we apply the basic CCA extending it to the elements in the lower triangular part, namely we keep on moving from left to right and top down but now visiting all the boxes in the matrix, in certain positions of the table we will have more than one unknown in the relevant inversion formula.

However, we can still manage by assuming these unknowns to be equal to each other, as we tacitly did earlier.

Let us sum up the CCA main advantages and typical problems.

1. *The correlation matrix is an exogenous input*;
2. The remaining inputs are *a complete swaption volatilities matrix and the zero coupon curve, so cap data are not involved in the calibration*;
3. *The calibration can be carried out through closed form formulas*;
4. If the industry formula is used for pricing swaptions in combination with Black's formula, *market swaption prices are recovered exactly*;

5. The method establishes *a one-to-one correspondence between model volatility parameters and market swaption volatilities*, at least in its basic form.

The last three points clearly represent the main advantages. The first point allows for imposing satisfactory instantaneous correlations.

Avoiding any optimization routine, CCA does not allow one to set any constraints on the output, so that there is no guarantee that the calibrated instantaneous volatilities will be real and non-negative. On the contrary, we have seen some cases in where we obtain negative entries in the output. We have solved this problem earlier by a rather drastic and too rough smoothing of the input swaption matrix.

Here we try and find different, less drastic ways to get rid of such inconveniences.

Cascade calibration: Further numerical studies

New input data of 1 feb 02, seen earlier. At first we will consider the results of calibration to only the upper (bold-faced) part of the swaption matrix.

The first exogenous correlation matrix we apply is Rebonato 3 parameters pivot, possibly rank-reduced. start with rank 7. The calibrated σ volatilities are

0.179									
0.153	0.155								
0.144	0.129	0.154							
0.144	0.134	0.105	0.156						
0.140	0.122	0.112	0.112	0.154					
0.143	0.134	0.103	0.101	0.106	0.153				
0.143	0.127	0.143	0.088	0.097	0.086	0.144			
0.146	0.153	0.128	0.078	0.070	0.098	0.093	0.145		
0.157	0.109	0.155	0.160	0.067	0.007	0.101	0.081	0.107	
0.136	0.152	0.126	0.123	0.121	0.108	-0.040	0.120	0.077	0.067

So there is a negative volatility, $\sigma_{10,7}$. What can we do to avoid this problem? Let us start by changing the rank of the correlation matrix. A calibration with full rank, equal to 19, gives us not only the same negative volatility, but also a complex one, $\sigma_{10,10}$.

Cascade calibration: Further numerical studies

Let us then try and reduce the rank. Down to rank 5 we get the same negative volatility, though reduced in absolute value. At rank 4 the negative entry disappears, and the output is completely acceptable, as visible in the following table.

0.179									
0.152	0.156								
0.131	0.130	0.165							
0.123	0.132	0.120	0.164						
0.128	0.123	0.120	0.118	0.153					
0.141	0.128	0.098	0.101	0.108	0.162				
0.144	0.115	0.122	0.082	0.102	0.106	0.159			
0.147	0.137	0.106	0.065	0.071	0.110	0.114	0.159		
0.156	0.098	0.136	0.131	0.054	0.031	0.119	0.111	0.139	
0.134	0.147	0.117	0.106	0.095	0.086	0.007	0.138	0.102	0.122

The same happens for rank 3 and 2. What might cause a similar behaviour? Recall that lowering the rank of a correlation matrix amounts to impose an oscillating tendency to the columns, that for very low ranks is represented by a sigmoid-like shape. Some features of the lower rank correlations seem to be better suited to these swaptions data. In particular, we might elicit that correlation matrices characterized by less steep initial decorrelation allow for acceptable results.

Cascade calibration: Further numerical studies

More evidence? Further tests with synthetic correlation matrices, whose essential features can be easily modified and controlled. Let us see how the calibrated volatilities change with ρ_∞ and β in $\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\exp[-\beta|i - j|], \ \ \beta \geq 0.$ The parameters are modified for the exogenous ρ at each calibration (same swaptions inputs) as follows:
a) $\rho_\infty = 0.5$, $\beta = 0.05$; b) Reduce ρ_∞ to 0; c) Set β to 0.2; d) Set ρ_∞ up to 0.5; e) Set β to 0.4; f) Take $\beta = 0.2$ and $\rho_\infty = 0.4$; g) $\rho_\infty = 0$, $\beta = 0.1$.

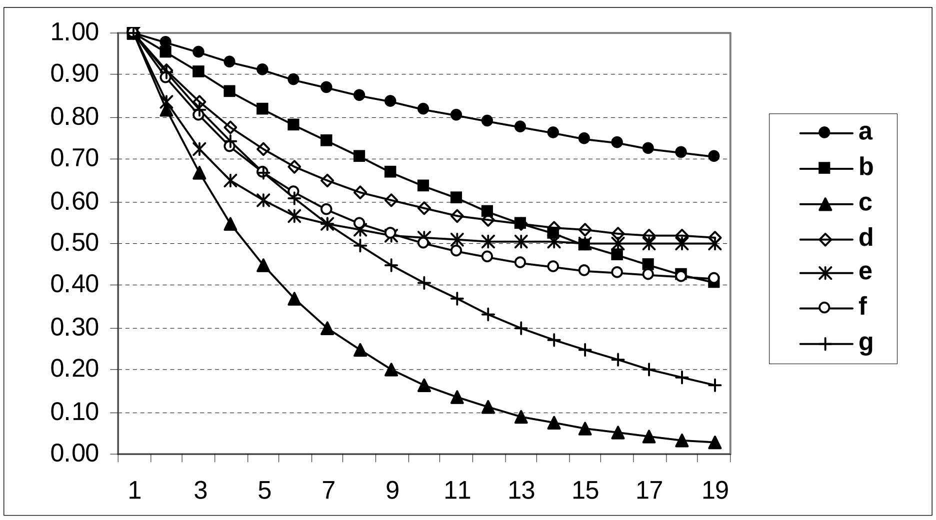


Figure 8: First columns of classic exponential structure for several values of the parameters

Cascade calibration: Further numerical studies

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty) \exp[-\beta|i - j|], \quad \beta \geq 0.$$

a) $\rho_\infty = 0.5$, $\beta = 0.05$; b) Reduce ρ_∞ to 0; c) Set β to 0.2; d) Set ρ_∞ up to 0.5; e) Set β to $0, 4$; f) Take $\beta = 0, 2$ and $\rho_\infty = 0.4$; g) $\rho_\infty = 0$, $\beta = 0, 1$.
We start with the matrix whose first column is represented by **a**, obtained by setting $\rho_\infty = 0.5$ and $\beta = 0.05$. With such a correlation, at full rank we obtain volatilities all real and positive, even calibrating to the entire swaption matrix. Then we lower the rank, first to 15 and then to 5, a level we keep in the following because representing the first problematic level when increasing the rank of Rebonato three-parameters form. We find acceptable results.

Then we move ρ_∞ and β, producing all the configurations shown, different in terms of extent of the decorrelation, initial steepness, and final level reached by correlation. With the correlations corresponding to **b**, **d** and **g**, we avoid negative or complex vols, whereas **c**, **e** and **f** give again a negative $\sigma_{10,7}$. We find bad results for those correlations featuring columns initially steeper, while the four configurations characterized by less initial steepness result in real and positive volatilities.

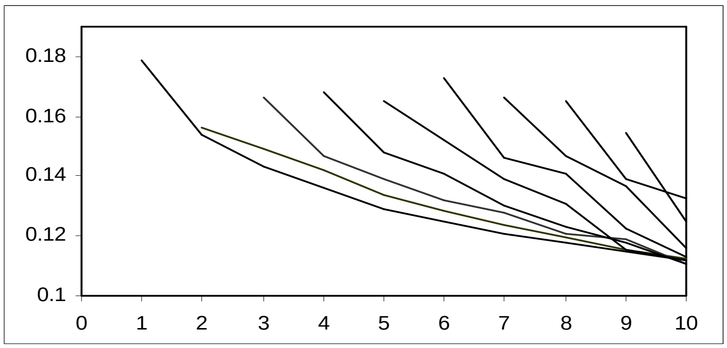
Cascade calibration: Further numerical studies

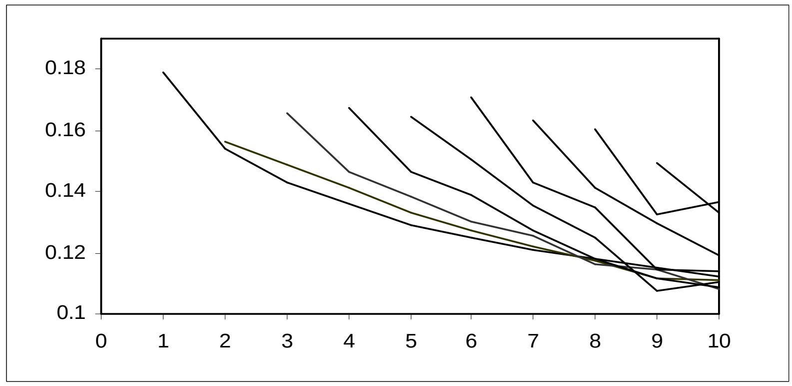
Now, let us see if S&C2 pivot can avoid problems for Rebonato 3 parameters correlations. S&C2 pivot is characterized by a more pronounced increase along sub-diagonals and less steep initial decorrelation. This correlation gives us volatilities all real and positive, at full 19 rank and when reducing the rank by optimizing a lower rank angles form onto the S&C2 pivot form. In particular, for rank 2 matrices, we do not have nonsensical correlations even if we calibrate to the entire matrix, as shown in the next table.

0.179									
0.152	0.156								
0.130	0.130	0.166							
0.119	0.131	0.122	0.167						
0.112	0.115	0.120	0.126	0.164					
0.112	0.115	0.100	0.113	0.126	0.171				
0.113	0.103	0.119	0.098	0.120	0.119	0.163			
0.122	0.124	0.108	0.082	0.091	0.121	0.119	0.160		
0.138	0.093	0.130	0.129	0.073	0.047	0.123	0.113	0.149	
0.121	0.129	0.106	0.098	0.092	0.090	0.023	0.144	0.118	0.147
0.120	0.120	0.101	0.093	0.134	0.063	0.060	0.045	0.142	0.108
0.107	0.107	0.107	0.142	0.036	0.135	0.078	0.063	0.051	0.143
0.112	0.112	0.112	0.112	0.084	0.084	0.074	0.108	0.062	0.052
0.103	0.103	0.103	0.103	0.103	0.123	0.116	0.043	0.105	0.061
0.097	0.097	0.097	0.097	0.097	0.097	0.169	0.088	0.068	0.108
0.093	0.093	0.093	0.093	0.093	0.093	0.093	0.153	0.117	0.089
0.094	0.094	0.094	0.094	0.094	0.094	0.094	0.094	0.090	0.155
0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.016
0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099

Cascade calibration: Further numerical studies

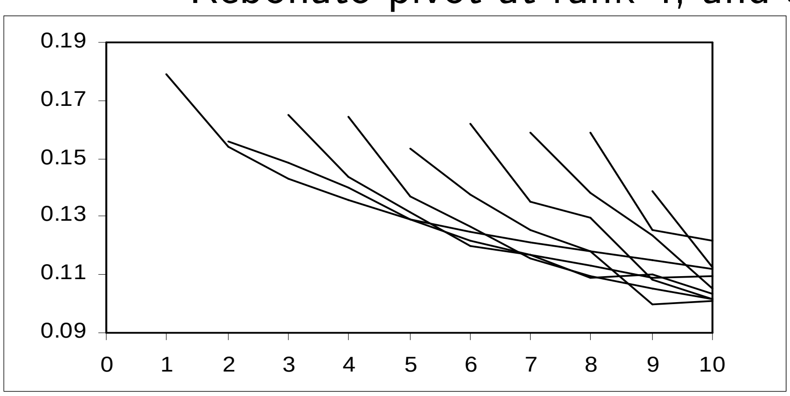
Diagnostics in these new cases? We examine first the evolution of the term structure of volatilities (TSV). We see below how it appears in case of a calibration with Rebonato three-parameters pivot correlation matrix at rank 2 (left) and with S&C2 at rank 2 (right).

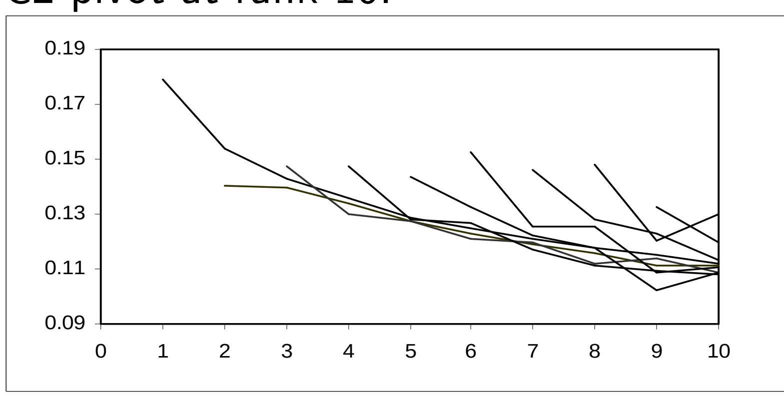




The left "Rebo3" evolution appears surprisingly regular, smooth and stable over time, as well as being rather realistic. The right "S&C2" evolution shows the same general features with a little worsening.

And when increasing the rank? We plot now the results with Rebonato pivot at rank 4, and S&C2 pivot at rank 10.





Cascade calibration: Further numerical studies

Now we examine terminal correlations (TC's). Low rank correlation matrices, through flat initial patterns, may induce oscillating TC patterns. Better with high rank.
10y TC with S&C2 pivot rank 2 and rank 10.

	10	11	12	13	14	15	16	17
10	1.000	0.928	0.895	0.920	0.855	0.846	0.928	0.924
11	0.928	1.000	0.863	0.909	0.933	0.881	0.901	0.923
12	0.895	0.863	1.000	0.916	0.908	0.910	0.878	0.939
13	0.920	0.909	0.916	1.000	0.944	0.931	0.956	0.926
14	0.855	0.933	0.908	0.944	1.000	0.954	0.923	0.928
15	0.846	0.881	0.910	0.931	0.954	1.000	0.937	0.958
16	0.928	0.901	0.878	0.956	0.923	0.937	1.000	0.957
17	0.924	0.923	0.939	0.926	0.928	0.958	0.957	1.000

	10	11	12	13	14	15	16	17
10	1.000	0.887	0.806	0.792	0.708	0.690	0.757	0.734
11	0.887	1.000	0.822	0.837	0.825	0.746	0.749	0.753
12	0.806	0.822	1.000	0.877	0.841	0.820	0.758	0.801
13	0.792	0.837	0.877	1.000	0.919	0.877	0.881	0.806
14	0.708	0.825	0.841	0.919	1.000	0.932	0.878	0.840
15	0.690	0.746	0.820	0.877	0.932	1.000	0.915	0.914
16	0.757	0.749	0.758	0.881	0.878	0.915	1.000	0.934
17	0.734	0.753	0.801	0.806	0.840	0.914	0.934	1.000

Low rank corr is OK for TSV; High rank corr is OK for TC; However, using particularly smooth and stylized corr it is possible to attain a regular evolution even at full rank.

Cascade calibration: Further numerical studies

Although one may find comfort in the existence of typical correlation features avoiding the common problems of cascade algorithms, it is worthwhile to keep in mind that such results depend on the particular market quotations we had available, and similar analysis should be carried out again for markedly different market situations. Moreover, we remark that intermediate configurations, with respect to the features we considered to be decisive, might give rise to less clear results, possibly due to the influence of some different, less evident factors. Finally, these findings depend also on the interpolation used for missing market quotations. We address this issue now.

Cascade calibration: Further numerical studies

In all previous cascade tests negative or complex σ's occur only for input swaptions **artificial** volatilities obtained by local linear interpolation along the columns of the swaption matrix.
On the contrary, volatilities obtained before such artificial interpolated values are all real and positive.
Let us check whether the linear interpolation is really the most suited for patterns in the swaption market. Following Morini (2002), fit a log-linear (or "power") functional form in the maturity to the matrix columns. For example, with our values, the fitted first column is

$$Y = 0.1785\,(X)^{-0.201}\,,\ \text{or}\ \ln(Y) = \ln(0.1785) - 0.201\,\ln(X),$$

where Y denotes the swaption volatility and X the maturity.

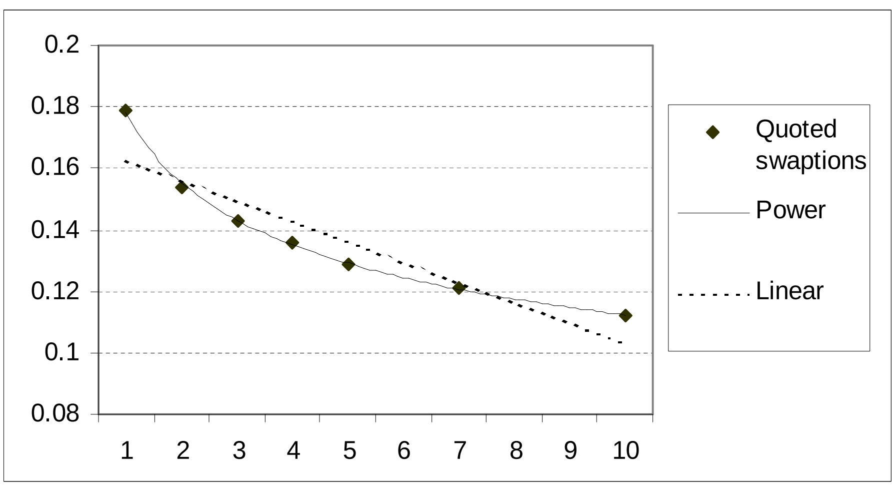


Figure 9: First columns of the swaptions data with fitted linear and log-linear parametric forms

Cascade calibration: Further numerical studies

The power fitting form appears clearly closer to the real market pattern than the linear one, as is further confirmed by standard diagnostics concerning the optimization output.

Also a graphical comparison regarding the other columns confirms the superiority of the power form. In order to make sure this was not a one-off coincidence, we tried the same with quotations referring to some months later, finding analogous results.

However, we must recall what is reported in Rebonato and Joshi (2001) about typical swaption configurations. According to this work, two are the common shape patterns that can be found in the Euro swaption market: a humped one, called *normal* and typical of periods of stability, and a monotonically decreasing one, called *excited* since associated with periods immediately following large movement in the yield curve and in the swaption matrix. Our data appear easily to belong to the second pattern. Of course, in periods characterized by humped patterns, a similar form would be likely to prove inadequate.

Cascade calibration: Further numerical studies

It is natural to wonder whether, using such a more realistic interpolation for missing maturities, it is possible to change the output of the cascade calibration. Keep now the original swaption matrix entries of february 1 except for the 7y row. Replace the 7y row by the fitted log-linear values and add the 6y, 8y and 9y maturity rows computed by this fitted form. Errors for the replaced 7th row are (upper part of the matrix, 4 entries)

Errors (differences)	-0.00028	-0.00119	-0.00079	0.00049
% Errors	-0.23272%	-1.03388%	-0.70780%	0.45776%

Keeping Rebonato three-parameters pivot as exogenous ρ, and calibrating to the upper part of the swaption matrix, **the previously found negative $\sigma_{10,7}$ disappears at any rank for the exogenous ρ. Even reaching full 19 rank, all volatilities are real and positive.**

This is not necessarily the solution, but shows that the choice of the interpolation technique is all but irrelevant!!

Endogenous Interpolation Cascade Calibration

A much more interesting step would the possibility to develop an analytical calibration **relying only on directly available market data** with no exogenous data interpolation.

We construct a new algorithm assuming σ parameters to be related in a pre-specified way when, due to the lack of market data needed to make a specific discernment, they surface as multiple unknowns. This way one can invert the industry swaption formula via cascade methods even in presence of "holes" in the market swaption matrix.

This method allows to have an exact consistent calibration based on *all* available market swaption quotes, and *only* on them. The new algorithm amounts to carrying out an endogenous interpolation, therefore it is called *Cascade Calibration with Endogenous Interpolation* Algorithm (*EICCA*).

Below we consider the simplest and most natural hypothesis on σ parameters, assuming the volatility of forward rates to be constant when no data are available to infer possible changes. We present the algorithm already extended for a complete calibration to the entire swaption matrix.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA)

(Rectangular EICCA (Morini (2003)).

1. Fix s, final dimension of the swaption matrix, and set

$$K := \{k \in \{0 \div s-1\} : v_{k,y} \text{ missing for } \quad y = k+1, \ldots, k+s\}$$

2. Set $\alpha = 0$;

3. a. If $\alpha \in K$, set $\sigma_{j,m+1} = \sigma_{j,m} = \ldots = \sigma_{j,\alpha+1} =: \sigma_j$ (*), $\alpha+1 \leq j \leq s$,, $m = \min\{i = \alpha+1, \ldots, s-1,\ i \notin K\}$. Set $\gamma = \alpha$ and $\alpha = m$.

   b. If $\alpha \notin K$, set $\gamma = \alpha$.

   Set $\beta = \alpha + 1$.

4. a. If $\gamma \in K$, solve the cascade 2nd order equation in σ_β with constraints (*).

   b. If $\gamma \notin K$, solve the cascade 2nd order equation in $\sigma_{\beta,\alpha+1}$.

5. Set $\beta = \beta + 1$. If $\beta < s + \gamma$ go to point 4. If $\beta = s + \gamma$, set $\sigma_{\beta,\alpha+1} = \sigma_{\beta,\alpha} = \ldots = \sigma_{\beta,1}$ and solve the cascade 2nd order equation in $\sigma_{\beta,\alpha+1}$. If $\beta < s + \alpha$, repeat point 5, else set $\alpha = \alpha + 1$.

6. If $\alpha < s$, go to point 3, else stop.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA)

Lest one should get confused by notation, notice that K is the set of indices for missing maturities, which obviously cannot include the last maturity considered, and m in point 3a) represents the index of the first market quoted maturity after missing maturity α.

When Algorithm 5 is applied to a typical Euro swaption matrix we have

$$K = \{5, 7, 8\},$$

namely the maturities at 6, 8 and 9 years after today. The algorithm determines all volatility parameters related to available swaptions, while correctly skipping the others.

For instance, with these missing maturities the volatility buckets $\sigma_{6,6}$, $\sigma_{8,8}$, $\sigma_{9,8}$ and $\sigma_{9,9}$ are not determined by the algorithm. In fact, notice that no market quoted swaption volatilities depend on them, and they do not affect the algorithm, which determines independently the other volatility buckets.

When needed, for example for presenting diagnostic structures, we use for these four buckets the homogeneity assumption $\sigma_{k,\beta(t)} =: \eta_{k-(\beta(t)-1)}$ getting $\sigma_{6,6} := \sigma_{5,5}$, $\sigma_{8,8} := \sigma_{7,7}$, $\sigma_{9,8} := \sigma_{8,7}$ and $\sigma_{9,9} := \sigma_{8,8}$.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA)

Having discarded the influence of exogenous artificial data, we can now check how cascade calibration really works on market data. We see below how algorithm EICC performs in practice.

As a first example, wee apply EICCA to previously used market data of February 1, 2002, with historically estimated correlation at full rank. This corresponds to one of the worst possible situations using basic CCA with exogenous artificial data, giving imaginary and negative entries in the upper triangular calibration considered, and many more if extending to the entire swaption matrix. With the new algorithm EICC results are:

Endogenous Interpolation Cascade Calibration Algorithm (EICCA)

0.179									
0.167	0.140								
0.153	0.138	0.138							
0.142	0.148	0.130	0.122						
0.135	0.131	0.134	0.135	0.109					
0.142	0.135	0.106	0.118	0.112	0.109				
0.155	0.126	0.145	0.098	0.130	0.087	0.087			
0.150	0.141	0.118	0.099	0.103	0.142	0.142	0.087		
0.130	0.092	0.136	0.153	0.095	0.122	0.122	0.142	0.087	
0.109	0.127	0.116	0.116	0.130	0.088	0.088	0.112	0.112	0.112
0.123	0.123	0.115	0.112	0.166	0.115	0.115	0.118	0.118	0.118
0.111	0.111	0.111	0.165	0.056	0.147	0.147	0.081	0.081	0.081
0.118	0.118	0.118	0.118	0.107	0.102	0.102	0.083	0.083	0.083
0.117	0.117	0.117	0.117	0.117	0.145	0.145	0.097	0.097	0.097
0.127	0.127	0.127	0.127	0.127	0.127	0.127	0.106	0.106	0.106
0.104	0.104	0.104	0.104	0.104	0.104	0.104	0.135	0.135	0.135
0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114
0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120
0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166

namely we have only real and positive σ's still allowing a perfect recovery of all market swaptions quotes.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA)

Considering earlier CCA tests, now with EICCA based only on market quotations all previously found numerical problems disappear, even for the previously highly problematic set of May 16, 2000 with its typical correlation matrix.

In addition data sets of February 1, 2002, December 10, 2002, and October 10, 2003, have been considered for general complete calibration testing, using as exogenous correlations the corresponding historically estimated matrices and their reduced rank versions. The historical estimations have been performed using one year of data prior to the trading day used for swaption data.

We considered in our tests reduced rank versions of all possible ranks from 2 to full rank 19. Results are summarized as follows.

Upper Triangular Calibration. This calibration was the typical reference case in the earlier CCA tests. Results included various anomalous results. Now with EICC no anomalous results or numerical problems have been found in any test outputs, at any correlation rank considered with any rank reduction method.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA)

Complete Rectangular Calibration. This calibration was almost always highly problematic with previous cascade calibration. Now, with EICCA, no anomalous results have been found in any test outputs, at any correlation rank with the eigenvalue zeroing by iteration rank reduction method (Morini and Webber, 2004).

Considering the angles parameterization rank reduction methodology seen above, results were analogously satisfactory with one single exception. For 2002 data, in the test with rank 4 correlation, we found two almost-zero negative volatilities, highly influenced by both homogeneity assumptions used, so that more realistic and flexible hypotheses could avoid them. But in practice it suffices to use the eigenvalue zeroing by iteration rank reduction technique, or S&C2 parametric form, to obtain positive σ's.

This exception is useful to notice that the fine details of volatility parameters have a precise dependence on the fine details of the correlation structure. Since usually instantaneous correlations are deemed not have a strong influence on swaption prices, this sensitivity can appear a flaw. On the other hand, it gives us a precise indication on the influence of instantaneous correlations on calibration, that with other methods can be hard to detect.

Cascade calibration: Further numerical studies

Possible integration of the Cascade Calibration with the cap market

The first point to address is the annualization of semi-annual caps data, so as to make them consistent with usually annual swaptions data. We have used the method in the earlier examples of joint calibration with ρ as calibration outputs.

Consider three instants $0 < S < T < U$, all six-months spaced, and assume we are dealing with an $S \times 1$ swaption and with S and T-expiry six-month caplets. Let us denote by v^2_{Black} the Black's swaption volatility and by $\sigma_1(t)$ and $\sigma_2(t)$, respectively, the instantaneous volatilities of the two semi-annual forward rates $F_1(t)$ and $F_2(t)$ associated with the two caplets, whereas $F(t)$ is the annual S-expiry forward rate. It is easy to derive the following approximate relationship to connect the above quantities:

$$v^2_{\text{Black}} \approx \frac{1}{S}\Big[u_1^2(0)\int_0^S \sigma_1(t)^2\,dt + u_2^2(0)\int_0^S \sigma_2(t)^2\,dt + 2\rho u_1(0)u_2(0)\int_0^S \sigma_1(t)\sigma_2(t)\,dt\Big],$$

$$u_{1,2}(t) = \frac{1}{F(t)}\left(\frac{F_{1,2}(t)}{2} + \frac{F_1(t)F_2(t)}{4}\right),$$

where ρ is the *infra-correlation* between the two semi-annual forward rates. When assuming constant inst vols, we have

$$v^2_{\text{Black}} \approx u_1^2(0)v^2_{S-\text{caplet}} + u_2^2(0)v^2_{T-\text{caplet}} + 2\rho u_1(0)u_2(0)v_{S-\text{caplet}}v_{T-\text{caplet}},$$

Cascade calibration: Further numerical studies

Given the last formulas, and setting infra-ρ's to 1, we can simply replace the first column of the input swaption matrix, containing volatilities for unitary length swaptions, with the corresponding array of annualized caplet volatilities. This is the method we used earlier for joint calibration. With the data of February 1 below

	Swaption volatilities	Semi-annual	rates	Caplet	volatilities
1	0,1790	0,0436	0,0480	0,1805	0,1720
2	0,1540	0,0483	0,0508	0,1911	0,1745
3	0,1430	0,0508	0,0523	0,1641	0,1575
4	0,1360	0,0532	0,0545	0,1546	0,1517
5	0,1290	0,0550	0,0560	0,1516	0,1480
6	0,1250	0,0559	0,0566	0,1445	0,1409
7	0,1210	0,0566	0,0572	0,1374	0,1352
8	0,1180	0,0560	0,0562	0,1329	0,1307
9	0,1150	0,0568	0,0571	0,1285	0,1262
10	0,1120	0,0575	0,0577	0,1240	0,1231

Table 4: Volatilities and forward rates on February 1, 2002

we obtain the annualized caplet vols

.178	.185	0.163	0.155	0.152	0.145	0.138	0.134	0.129	.125

Cascade calibration: Further numerical studies

Except for the first one, these values are all higher than the swaption volatilities they are to replace.

Replacing the corresponding Sx1 swaptions vols with these annualized cap vols and using exogenous Rebonato 3 parameters pivot correlation at rank four (most standard situation, used earlier), with a cascade calibration we obtain all real and positive σ's, with diagnostics similar to the first cascade calibration tests we performed with year 2000 data. In particular, we have a rapidly increasing TSV.

Assuming again constant inst vols and implying instead ρ's from both caplets and swaptions data, by inverting

$$v_{\text{Black}}^2 \approx u_1^2(0) v_{S-\text{caplet}}^2 + u_2^2(0) v_{T-\text{caplet}}^2 + 2\rho u_1(0) u_2(0) v_{S-\text{caplet}} v_{T-\text{caplet}},$$

we get

1.022	0.388	.543	.536	.444	.493	.533	.56	.586	.598

Cascade calibration: Further numerical studies

1.022	0.388	.543	.536	.444	.493	.533	.56	.586	.598

Besides the fact that the first value is outside the viable range for correlations, the other values appear too low to represent real correlations between adjacent rates. A possible reason for this is the aforementioned bias due to the chosen volatility parameterization. Again, more realistic hypothesis can lead to different results.
But the really relevant reasons calling for a cautious interpretation of such results are of a different nature. Indeed, relations and discrepancies between caps and swaptions tend to be influenced by causes concerning the market fundamentals. Does there exist a basic congruence between the cap and swaption markets, that a model can successfully detect and incorporate? Rebonato (2001) seems to warn against excessive enthusiasm in considering such a possibility. Rebonato recalls that problems such as illiquidities, agency problems and value-at-risk based limit structures strongly reduce the effectiveness of the *quasi-arbitrageurs* who are supposed to maintain the internal consistency between the two markets.

Accordingly, simple artificial values such as the infra-correlations above are likely to be actually influenced by many different external factors that are hard to detect and measure.

Cascade calibration: Conclusions

We remarked that some fundamental features make the Cascade methodology particularly appealing:
it is automatic and analytical, and hence instantaneous;
if a common industry approximation is used for pricing, it is free from any calibration error;
it allows for a direct correspondence between market swaption volatilities and LIBOR volatility parameters.
We pointed out that a further opportunity is given by the exogenous nature of the forward rates correlation matrix. Accordingly, we both calibrated with an exogenous historically estimated correlation matrix and considered regular and parsimonious parameterizations, being led to a simple and intuitive methodology to fix parameters consistently with general market tendencies. In this way instantaneous correlation matrices that are rather realistic, regular and simple to control and modify can be easily obtained. Moreover, as we showed, regular terminal correlations and a satisfactory evolution of the term structure of volatilities are possible, even though our tests revealed a possible trade-off between regularity of the evolution of the TSV and realism of TC's, depending on the level of the rank in the exogenous correlation matrix.
Further, we have given suggestions on the choice of the exogenous correlation matrix and on the interpolation technique for the swaption matrix that avoid negative or complex σ's.