

# **LIBOR and swap market models**

## **Lectures for the Fixed Income course**

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These lectures are mostly based on the book  
“Interest-Rate Models: Theory and Practice”,  
Springer Verlag, 2001

(by D. Brigo and F. Mercurio), to which we refer  
for a rigorous treatment and references.

See also the papers listed in the references at the end

# LIBOR AND SWAP MARKET MODELS.

## Outline.

### ● LECTURE 1

- Preliminaries: Risk Neutral Valuation;
- When a dynamics is not needed in valuation (FRA's, IRS's);
- Definition of forward LIBOR and SWAP rates;
- When a dynamics is needed (caps, swaptions) what do we model? Short rate? Instantaneous fwd rates? Market rates?
- Speaking the traders language: The importance of compatibility with Black's formulae for caps and swaptions
- Defs and Compatibility of the LIBOR model with Black's cap formula;
- Change of Numeraire: Rigorous derivation of Black's Caplet formula with the LIBOR model dynamics;
- Change of Numeraire: Derivation of the dynamics of forward LIBOR rates under different forward measures;
- Forward LIBOR rates dynamics under the spot LIBOR measure.

- **LECTURE 2**

- The swap market model;
- Change of Numeraire: rigorous derivation of Black's market formula for swaptions;
- Incompatibility between LIBOR and SWAP models;
- Parameterizing the LIBOR model: Instantaneous volatilities;
- Diagnostics after calibration: Term structure of caplet volatilitites and Terminal correlations;
- Instantaneous correlations: Some full rank parameterizations.
- Instantaneous correlations: Reduced rank parameterizations.
- Monte Carlo pricing with the LIBOR model
- An approximated swaption formula linking the LIBOR model to the swaption market restoring mutual compatibility in practice
- Derivation of the formula for terminal correlation;

- **LECTURE 3**

- Calibration: The data.
- Calibration: Instantaneous correlations as inputs or outputs?
- Instantaneous correlations: Calibration Inputs? Historical estimation of correlation matrices and Pivot forms
- Instantaneous correlations: Calibration Outputs? Calibration examples to Caps and Swaptions with diagnostics;
- Instantaneous correlations: Calibration Inputs? Swaptions exact analytical cascade calibration: Practical examples of calibration and diagnostics.
- Advanced Cascade Calibration: impact of the chosen parameterization for instantaneous correlation and of the interpolation technique for missing/illiquid input data

- **LECTURE 4**

- Pricing with the LIBOR model:
- Libor in Arrears;
- Constant Maturity Swaps (CMS);
- Ratchet caps and floors;
- Zero Coupon Swaptions.

# LECTURE 1.

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## Zero-coupon Bond, LIBOR rate

Bank account  $dB(t) = r_t B(t) dt$ ,  $B(t) = B_0 \exp\left(\int_0^t r_s ds\right)$ .  
Risk neutral measure  $Q$  associated with numeraire  $B$ ,  $Q = Q^B$ .

Recall shortly the risk-neutral valuation paradigm of Harrison and Pliska's (1983), characterizing no-arbitrage theory:

A future stochastic payoff  $H_T$ , built on an underlying fundamental asset, paid at a future time  $T$  and satisfying some technical conditions, has as unique price at current time  $t$  the *risk neutral world* expectation

$$E_t^B \left[ \frac{B(t)}{B(T)} H_{(\text{Asset})_T} \right] = E_t^Q \left[ \exp\left(-\int_t^T r_s ds\right) H_{(\text{Asset})_T} \right]$$

## Risk neutral valuation

$$E_t^B \left[ \frac{B(t)}{B(T)} H(\text{Asset})_T \right] = E_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) H(\text{Asset})_T \right]$$

“Risk neutral world” means that all fundamental underlying assets must have as locally deterministic drift rate the risk-free interest rate  $r$ :

$$d \text{Asset}_t = \boxed{r_t} \text{Asset}_t dt + \\ + \text{Asset-Volatility}_t (d \text{Brownian-motion-under-Q})_t$$

Nothing strange at first sight. To value **future unknown** quantities now, we discount at the relevant interest rate and then take **expectation**.

The mean is a reasonable estimate of unknown quantities with known distributions. But what is surprising is that we do not take the mean in the **real world**, where statistics and econometrics based on the observed data are used. Indeed, in the real world probability measure  $P$ , we have

$$d \text{Asset}_t = \boxed{\mu_t} \text{Asset}_t dt + \\ + \text{Asset-Volatility}_t (d \text{Brownian-motion-under-P})_t.$$

## Risk-neutral valuation

But when we consider risk-neutral valuation, or no-arbitrage pricing, we do not use the real-world  $P$ -dynamics with  $\mu$  but rather the risk-neutral world  $Q$ -dynamics with  $r$ . So in

$$E_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) H(\text{Asset})_T \right] = E_t^B \left[ \frac{B(t)}{B(T)} H(\text{Asset})_T \right]$$

what might look strange at first sight is that  $\mu$ , i.e. **the growth rate of our asset** (e.g. a stock) **in the real world does not enter the price.**

**Even if two investors do not agree on the expected return of a fundamental asset in the real world, they still agree on the price of derivatives (e.g. options) built on this asset.**

This is the reason for the enormous success of Option pricing theory, and partly for the Nobel award to Black, Scholes and Merton who started it. According to Stephen Ross (MIT) in the Palgrave Dictionary of Economics:

**”... options pricing theory is the most successful theory not only in finance, but in all of economics”.**

From the risk neutral valuation formula we see that one fundamental quantity is  $r_t$ , the instantaneous interest rate. In particular, if we take  $H_T = 1$ , we obtain the Zero-Coupon Bond

## Zero-coupon Bond, LIBOR rate

A  **$T$ -maturity zero-coupon bond** is a contract which guarantees the payment of one unit of currency at time  $T$ . The contract value at time  $t < T$  is denoted by  $P(t, T)$ :

$$P(T, T) = 1,$$

$$P(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)} 1 \right] = E_t^Q \exp \left( - \int_t^T r_s ds \right) = E_t^Q D(t, T)$$

All kind of rates can be expressed in terms of zero-coupon bonds and vice-versa. ZCB's can be used as fundamental quantities.

The **spot-Libor rate** at time  $t$  for the maturity  $T$  is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from  $P(t, T)$  units of currency at time  $t$ , when accruing occurs **proportionally** to the investment time.

$$P(t, T)(1 + (T-t) L(t, T)) = 1, \quad L(t, T) = \frac{1 - P(t, T)}{(T-t) P(t, T)} .$$

Notice:

$$r(t) = \lim_{T \rightarrow t^+} L(t, T) \approx L(t, t + \epsilon),$$

$\epsilon$  small.

## LIBOR rate, zero coupon curve (term structure of interest rates)

The **zero-coupon curve** (often referred to as “yield curve” or “term structure”) at time  $t$  is the graph of the function

$$T \mapsto L(t, T), \quad \text{initial point } r_t \approx L(t, t + \epsilon).$$

This function is called *term structure of interest rates* at time  $t$ .

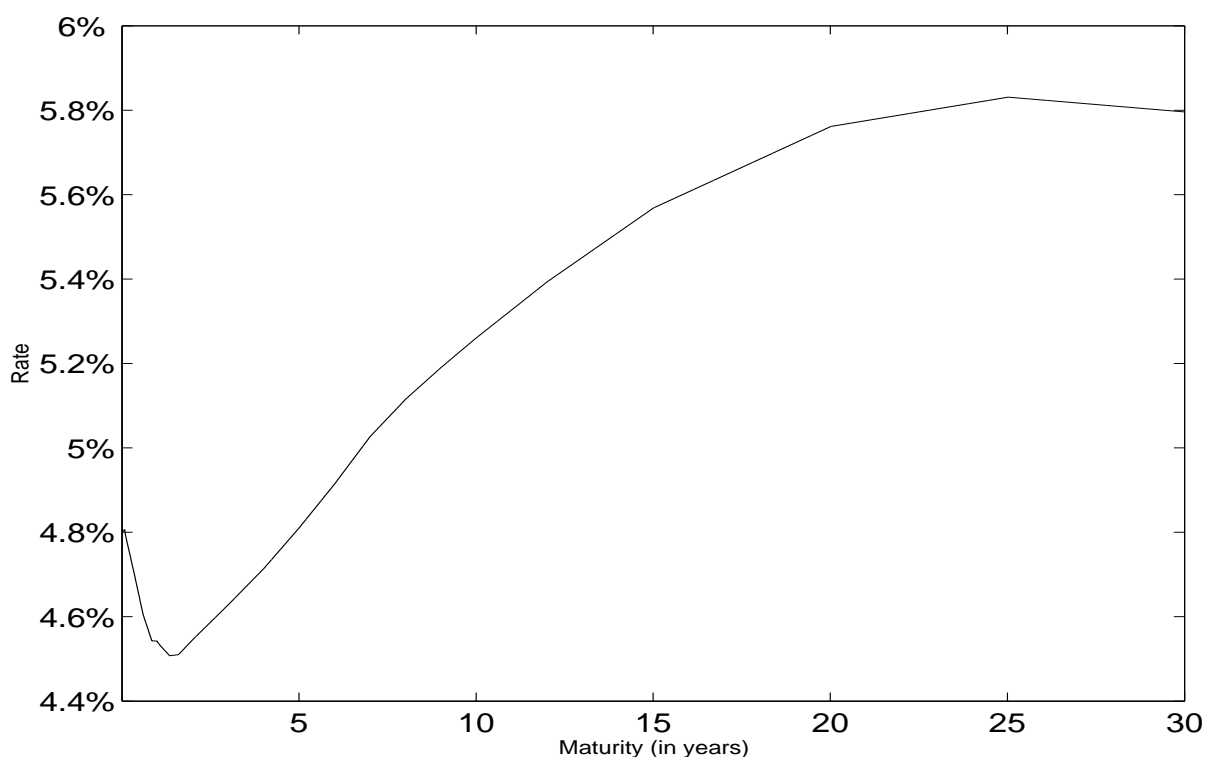


Figure 1: Zero-coupon curve  $T \mapsto L(t, t + T)$  stripped from market EURO rates on February 13, 2001,  $T = 0, \dots, 30y$

## Products not depending on the curve dynamics: FRA's and IRS's

At time  $S$ , with reset time  $T$  ( $S > T$ )

Fixed payment

$$\longrightarrow (S - T)K \longrightarrow$$

$$\longleftarrow (S - T) L(T, S) \longleftarrow$$

Float. payment

A **forward rate agreement** FRA is a contract involving three time instants: The current time  $t$ , the expiry time  $T > t$ , and the maturity time  $S > T$ . The contract gives its holder an interest rate payment for the period  $T \mapsto S$  with fixed rate  $K$  at maturity  $S$  against an interest rate payment over the same period with rate  $L(T, S)$ .

Basically, this contract allows one to lock-in the interest rate between  $T$  and  $S$  at a desired value  $K$ .

By easy static no-arbitrage arguments:

$$\text{FRA}(t, T, S, K) = P(t, S)(S - T)K - P(t, T) + P(t, S) .$$

$(S - T)$  may be replaced by a year fraction  $\tau$ . The value of  $K$  which makes the contract fair ( $=0$ ) is the **forward LIBOR interest rate** prevailing at time  $t$  for the expiry  $T$  and maturity  $S$ :  $K = F(t; T, S)$ .

$$F(t; T, S) := \frac{1}{S - T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right) .$$

## Products not depending on the curve dynamics: FRA's and IRS's

An Interest Rate Swap (PFS) is a contract that exchanges payments between two differently indexed legs, starting from a future time-instant. At future dates  $T_{\alpha+1}, \dots, T_{\beta}$ ,

$$\begin{array}{ccc}
 & \longrightarrow & \tau_j K & \longrightarrow \\
 \text{at } T_j : & \text{Fixed Leg} & & \text{Float. Leg} \\
 & \longleftarrow & \tau_j L(T_{j-1}, T_j) & \longleftarrow \\
 & & \tau_j F(T_{\alpha}; T_{j-1}, T_j) & 
 \end{array}$$

where  $\tau_i = T_i - T_{i-1}$ . The IRS is called “payer IRS” from the company paying  $K$  and “receiver IRS” from the company receiving  $K$ .

The *discounted* payoff at a time  $t < T_{\alpha}$  of a receiver IRS is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) \tau_i (K - L(T_{i-1}, T_i)), \quad \text{or alternatively}$$

$$D(t, T_{\alpha}) \sum_{i=\alpha+1}^{\beta} P(T_{\alpha}, T_i) \tau_i (K - F(T_{\alpha}; T_{i-1}, T_i)).$$

IRS can be valued as a collection of FRAs.

## Products not depending on the curve dynamics: FRA's and IRS's

A receiver IRS can be valued as a collection of (receiver) FRAs.

$$\begin{aligned} \text{ReceiverIRS}(t, [T_\alpha, \dots, T_\beta], K) &= \\ &= \sum_{i=\alpha+1}^{\beta} P(t, T_i) \tau_i (K - F(t; T_{i-1}, T_i)), \quad \text{or alternatively} \\ &= \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) - P(t, T_\alpha) + P(t, T_\beta). \end{aligned}$$

Analogously,

$$\begin{aligned} \text{PayerIRS}(t, [T_\alpha, \dots, T_\beta], K) &= \\ &= \sum_{i=\alpha+1}^{\beta} P(t, T_i) \tau_i (F(t; T_{i-1}, T_i) - K), \quad \text{or alternatively} \\ &= - \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) + P(t, T_\alpha) - P(t, T_\beta). \end{aligned}$$

The value  $K = S_{\alpha, \beta}(t)$  which makes  $\text{IRS}(t, [T_\alpha, \dots, T_\beta], K) = 0$  is the **forward swap rate**.

Denote  $F_i(t) := F(t; T_{i-1}, T_i)$ .

## Products not depending on the curve dynamics: FRA's and IRS's

Three possible formulas for the forward swap rate:

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t), \quad w_i(t) = \frac{\alpha_i P(t, T_i)}{\sum_{j=\alpha+1}^{\beta} \alpha_j P(t, T_j)}$$

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1 + \tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^i \frac{1}{1 + \tau_j F_j(t)}} .$$

The second expression is a “weighted” average:  $0 \leq w_i \leq 1$ ,  $\sum_{j=\alpha+1}^{\beta} w_j = 1$ . The weights are functions of the  $F$ 's and thus random at future times.

## Products not depending on the curve dynamics: FRA's and IRS's

Recall the Receiver IRS Formula

$$\begin{aligned} \text{ReceiverIRS}(t, [T_\alpha, \dots, T_\beta], K) &= \\ &= \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) - P(t, T_\alpha) + P(t, T_\beta) \end{aligned}$$

and combine it with

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

to obtain

$$\begin{aligned} \text{ReceiverIRS}(t, [T_\alpha, \dots, T_\beta], K) &= \\ &= (K - S_{\alpha,\beta}(t)) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \end{aligned}$$

Analogously,

$$\begin{aligned} \text{PayerIRS}(t, [T_\alpha, \dots, T_\beta], K) &= \\ &= (S_{\alpha,\beta}(t) - K) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \end{aligned}$$

## Products depending on the curve dynamics: Caplets and CAPS

A **cap** can be seen as a payer IRS where each exchange payment is executed only if it has positive value.

$$\begin{aligned} \text{Cap discounted payoff: } & \sum_{i=\alpha+1}^{\beta} D(t, T_i) \tau_i (L(T_{i-1}, T_i) - K)^+ . \\ & = \sum_{i=\alpha+1}^{\beta} D(t, T_i) \tau_i (F_i(T_{i-1}) - K)^+ . \end{aligned}$$

Suppose a company is Libor–indebted and has to pay at  $T_{\alpha+1}, \dots, T_{\beta}$  the Libor rates resetting at  $T_{\alpha}, \dots, T_{\beta-1}$ . The company has a view that libor rates will increase in the future, and wishes to protect itself

$$\text{buy a cap: } (L - K)^+ \xrightarrow{CAP} \text{Company} \xrightarrow{DEBT} L$$

$$\text{or Company} \xrightarrow{NET} L - (L - K)^+ = \min(L, K)$$

The company pays at most  $K$  at each payment date.

## Products depending on the curve dynamics: CAPS

A cap contract can be decomposed additively: Indeed, the discounted payoff is a sum of terms (**caplets**)

$$\begin{aligned} & D(t, T_i) \tau_i (L(T_{i-1}, T_i) - K)^+ \\ &= D(t, T_i) \tau_i (F_i(T_{i-1}) - K)^+ . \end{aligned}$$

Each caplet can be evaluated separately, and the corresponding values can be added to obtain the cap price (notice the “call option” structure!).

However, even if separable, the payoff is not linear in the rates. This implies that, roughly speaking, we need the whole distribution of future rates, and not just their means, to value caplets. This means that the dynamics of interest rates is needed to value caplets: We cannot value caplets at time  $t$  based only on the current zero curve  $T \mapsto L(t, T)$ , but we need to specify how this infinite-dimensional object moves, in order to have its distribution at future times. This can be done for example by specifying how  $r$  moves.

## Products depending on the curve dynamics: SWAPTIONS

Finally, we introduce options on IRS's (**swaptions**).

A (payer) swaption is a contract giving the right to enter at a future time a (payer) IRS.

The time of possible entrance is the maturity.

Usually maturity is first reset of underlying IRS.

IRS value at its first reset date  $T_\alpha$ , i.e. at maturity, is, by our above formulas,

$$\begin{aligned} \text{PayerIRS}(T_\alpha, [T_\alpha, \dots, T_\beta], K) &= \\ &= \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) \tau_i (F(T_\alpha; T_{i-1}, T_i) - K) = \\ &= (S_{\alpha,\beta}(T_\alpha) - K) \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \end{aligned}$$

Call  $C_{\alpha,\beta}(T_\alpha)$  the summation on the right hand side.

## Products depending on the curve dynamics: SWAPTIONS

The option will be exercised only if this IRS value is positive. There results the payer–swaption discounted–payoff at time  $t$ :

$$D(t, T_\alpha) C_{\alpha, \beta}(T_\alpha) (S_{\alpha, \beta}(T_\alpha) - K)^+ =$$

$$D(t, T_\alpha) \left( \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) \tau_i (F(T_\alpha; T_{i-1}, T_i) - K) \right)^+ .$$

Unlike Caps, this payoff **cannot be decomposed** additively.

Caps can be decomposed in caplets, each with a single fwd rate. Caps: Deal with each caplet **separately**, and put results together.

Only **marginal** distributions of different fwd rates are involved.

Not so with swaptions: The summation is *inside* the positive part operator  $()^+$ , and not outside.

With swaptions we will need to consider the *joint* action of the rates involved in the contract.

The **correlation** between rates is fundamental in handling swaptions, contrary to the cap case.

## Which variables do we model?

For some products (Forward Rate Agreements, Interest Rate Swaps) the **dynamics** of interest rates is not necessary for valuation, the current curve being enough.

For caps, swaptions and more complex derivatives a dynamics is necessary.

Specifying a stochastic dynamics for interest rates amounts to choosing an **interest-rate model**.

- Which quantities do we model? Short rate  $r_t$ ? LIBOR rates  $L(t, T)$ ? Forward LIBOR rates  $F_i(t) = F(t; T_{i-1}, T_i)$ ? Forward Swap rates  $S_{\alpha, \beta}(t)$ ? Bond Prices  $P(t, T)$ ?
- How is randomness modeled? i.e: What kind of stochastic process or stochastic differential equation do we select for our model? (Markov diffusions)
- What are the consequences of our choice in terms of valuation of market products, ease of implementation, goodness of calibration to real data, pricing complicated products with the calibrated model, possibilities for diagnostics on the model outputs and implications, stability, robustness, etc?

## First Choice: short rate $r$

This approach is based on the fact that the zero coupon curve at any instant, or the (informationally equivalent) zero bond curve

$$T \mapsto P(t, T) = E_t^Q \exp \left( - \int_t^T \boxed{r_s} ds \right)$$

is completely characterized by the probabilistic/dynamical properties of  $r$ . So we write a model for  $r$ , the initial point of the curve  $T \mapsto L(t, T)$  for  $T = t$  at every instant  $t$ .

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t$$

- Unrealistic correlation patterns between points of the curve with different maturities. for example, in one-factor short-rate models

$$\text{Corr}(dF_i(t), dF_j(t)) = 1;$$

- Poor calibration capabilities: can only fit a low number of caps and swaptions unless dangerous and uncontrollable extensions are taken into account;
- Difficulties in expressing market views and quotes in terms of model parameters;
- Related lack of agreement with market valuation formulas for basic derivatives.
- Models that are good as distribution (lognormal models) are not analytically tractable and have problems of explosion for the bank account.

## What do we model? Second Choice: instantaneous forward rates $f(t, T)$

Recall the forward LIBOR rate at time  $t$  between  $T$  and  $S$ ,  $F(t; T, S) = (P(t, T)/P(t, S) - 1)/(S - T)$ , which makes the FRA contract to lock in at time  $t$  interest rates between  $T$  and  $S$  fair. When  $S$  collapses to  $T$  we obtain *instantaneous* forward rates:

$$f(t, T) = \lim_{S \rightarrow T^+} F(t; T, S) = -\frac{\partial \ln P(t, T)}{\partial T}, \quad \lim_{T \rightarrow t} f(t, T) = r_t.$$

Why should one be willing to model the  $f$ 's at all? The  $f$ 's are not observed in the market, so that there is no improvement with respect to modeling  $r$  in this respect. Moreover notice that  $f$ 's are more structured quantities:

$$f(t, T) = -\frac{\partial \ln E_t \left[ \exp \left( - \int_t^T r(s) ds \right) \right]}{\partial T},$$

$$P(t, T) = e^{-\int_t^T f(t, u) du}$$

Given the structure in  $r$ , we may expect some restrictions on the risk-neutral dynamics that are allowed for  $f$ .

## What do we model? Second Choice: instantaneous forward rates $f(t, T)$

Indeed, there is a fundamental theoretical result: Set  $f(0, T) = f^M(0, T)$ . We have

$$df(t, T) = \boxed{\sigma(t, T) \left( \int_t^T \sigma(t, s) ds \right)} dt + \sigma(t, T) dW(t),$$

under the risk neutral world measure, if no arbitrage has to hold. Thus we find that the no-arbitrage property of interest rates dynamics is here clearly expressed as a **link between the local standard deviation (volatility or diffusion coefficient) and the local mean (drift)** in the dynamics. Given the volatility, there is no freedom in selecting the drift, contrary to the more fundamental models based on  $dr_t$ , where the whole risk neutral dynamics was free:

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t$$

$b$  and  $\sigma$  had no link due to no-arbitrage.

## Second Choice, modeling $f$ (HJM): is it worth it?

$$df(t, T) = \boxed{\sigma(t, T) \left( \int_t^T \sigma(t, s) ds \right)} dt + \sigma(t, T) dW(t),$$

This can be useful to study arbitrage free properties of models, but when in need of writing a concrete model to price and hedge financial products, all useful models coming out of HJM are the already known short rate models seen earlier (these are particular HJM models, especially Gaussian models) or the market models we are going to see later.

Even though market models do not necessarily need the HJM framework to be derived, HJM may serve as a unifying framework in which all categories of no-arbitrage interest-rate models can be expressed.

## What do we model? Third choice: Market rates. The MARKET MODELS. Guided Tour

Before market models were introduced, short-rate models used to be the main choice for pricing and hedging interest-rate derivatives.

Short-rate models are still chosen for many applications and are based on modeling the instantaneous spot interest rate (“short rate”  $r_t$ ) via a (possibly multi-dimensional) diffusion process. This diffusion process characterizes the evolution of the complete yield curve in time.

To introduce market models, recall the forward LIBOR rate at time  $t$  between  $T$  and  $S$ ,

$$F(t; T, S) = \frac{1}{(S - T)}(P(t, T)/P(t, S) - 1),$$

which makes the FRA contract to lock in at time  $t$  interest rates between  $T$  and  $S$  fair ( $=0$ ). **A family of such rates for  $(T, S) = (T_{i-1}, T_i)$  spanning  $T_0, T_1, T_2, \dots, T_M$  is modeled in the LIBOR market model. These are rates associated to market payoffs (FRA's) and not abstract rates such as  $r_t$  or  $f(t, T)$  (rates on infinitesimal maturities/tenors).**

## Third Choice: Market models. Guided Tour

To further motivate market models, let us consider the time-0 price of a  $T_2$ -maturity caplet resetting at time  $T_1$  ( $0 < T_1 < T_2$ ) with strike  $X$  and a notional amount of 1. Let  $\tau$  denote the year fraction between  $T_1$  and  $T_2$ . Such a contract pays out at time  $T_2$  the amount

$$\tau(L(T_1, T_2) - X)^+ = \tau(F_2(T_1) - X)^+.$$

On the other hand, the market has been pricing caplets (actually caps) with Black's formula for years. Let us see how this formula is rigorously derived under the LIBOR model dynamics, the only dynamical model that is consistent with it.

**FACT ONE.** *The price of any asset divided by a reference asset (called numeraire) is a martingale (no drift) under the measure associated with that numeraire.*

In particular,

$$F_2(t) = \frac{(P(t, T_1) - P(t, T_2))/(T_2 - T_1)}{P(t, T_2)},$$

is a portfolio of two zero coupon bonds divided by the zero coupon bond  $P(\cdot, T_2)$ . If we take the measure  $Q^2$  associated with the numeraire  $P(\cdot, T_2)$ , by FACT ONE  $F_2$  will be a martingale (no drift) under that measure.

## Third Choice: Market models. Guided Tour

$F_2$  is a martingale (no drift) under that  $Q^2$  measure associated with numeraire  $P(\cdot, T_2)$ .

FACT TWO: THE TIME- $t$  RISK NEUTRAL PRICE

$$\text{Price}_t = E_t^{\boxed{B}} \left[ \frac{\boxed{B(t)} \text{Payoff}(T)}{\boxed{B(T)}} \right]$$

IS INVARIANT BY CHANGE OF NUMERAIRE: IF  $S$  IS ANY OTHER NUMERAIRE, WE HAVE

$$\text{Price}_t = E_t^{\boxed{S}} \left[ \frac{\boxed{S_t} \text{Payoff}(T)}{\boxed{S_T}} \right].$$

IN OTHER TERMS, IF WE SUBSTITUTE THE THREE OCCURRENCES OF THE NUMERAIRE WITH A NEW NUMERAIRE THE PRICE DOES NOT CHANGE.

## Third Choice: Market models. Guided Tour

Consider now the caplet price and apply FACT TWO: Replace  $B$  with  $P(\cdot, 2)$

$$\begin{aligned} E^B \left[ \frac{B(0)}{B(T_2)} \tau (F_2(T_1) - X)^+ \right] &= \\ &= E^{Q^2} \left[ \frac{P(0, T_2)}{P(T_2, T_2)} \tau (F_2(T_1) - X)^+ \right] \end{aligned}$$

Take out  $P(0, T_2)$  and recall that  $P(T_2, T_2) = 1$ . We have

$$= P(0, T_2) E^{Q^2} \tau \left[ (F_2(T_1) - X)^+, \right]$$

By fact ONE  $F_2$  is a martingale (no drift) under  $Q_2$ . Take a geometric Brownian motion

$$dF(t; T_1, T_2) = \boxed{\sigma_2(t)} F(t; T_1, T_2) dW_2(t), \quad \text{mkt } F(0; T_1, T_2)$$

where  $\sigma_2$  is the instantaneous volatility, assumed here to be constant for simplicity, and  $W_2$  is a standard Brownian motion under the measure  $Q^2$ . **The forward LIBOR rates  $F$ 's are the quantities that are modeled instead of  $r$  and  $f$  in the LIBOR market model.**

## Third Choice: Market models. Guided Tour

$$dF_2(t) = \boxed{\sigma_2(t)} F_2(t) dW_2(t), \quad \text{mkt } F_2(0)$$

Let us solve this equation and compute  $E^{Q^2} [(F_2(T_1) - X)^+, ]$ .  
By Ito's formula:

$$\begin{aligned} d \ln(F_2(t)) &= \ln'(F_2) dF_2 + \frac{1}{2} \ln''(F_2) dF_2 dF_2 \\ &= \frac{1}{F_2} dF_2 + \frac{1}{2} \left( -\frac{1}{(F_2)^2} \right) dF_2 dF_2 = \\ &= \frac{1}{F_2} \sigma_2 F_2 dW_2 - \frac{1}{2} \frac{1}{(F_2)^2} (\sigma_2 F_2 dW_2) (\sigma_2 F_2 dW_2) = \\ &= \sigma_2 dW_2 - \frac{1}{2} \frac{1}{(F_2)^2} \sigma_2^2 F_2^2 dW_2 dW_2 = \\ &= \sigma_2(t) dW_2(t) - \frac{1}{2} \sigma_2^2(t) dt \end{aligned}$$

(we used  $dW_2 dW_2 = dt$ ). So we have

$$d \ln(F_2(t)) = \sigma_2(t) dW_2(t) - \frac{1}{2} \sigma_2^2(t) dt$$

## Third Choice: Market models. Guided Tour

So we have

$$d \ln(F_2(t)) = \sigma_2(t) dW_2(t) - \frac{1}{2} \sigma_2^2(t) dt$$

Integrate both sides:

$$\int_0^T d \ln(F_2(t)) = \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt$$

$$\ln(F_2(T)) - \ln(F_2(0)) = \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt$$

$$\ln \frac{F_2(T)}{F_2(0)} = \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt$$

$$\frac{F_2(T)}{F_2(0)} = \exp \left( \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt \right)$$

$$F_2(T) = F_2(0) \exp \left( \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt \right)$$

## Third Choice: Market models. Guided Tour

$$F_2(T) = F_2(0) \exp \left( \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt \right)$$

Compute the distribution of the random variable in the exponent.

It is Gaussian, since it is a stochastic integral of a deterministic function times a Brownian motion (sum of independent Gaussians is Gaussian).

## Third Choice: Market models. Guided Tour

Compute the expectation:

$$E\left[\int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt\right] = 0 - \frac{1}{2} \int_0^T \sigma_2^2(t) dt$$

and the variance

$$\begin{aligned} \text{Var} \left[ \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt \right] &= \\ &= \text{Var} \left[ \int_0^T \sigma_2(t) dW_2(t) \right] \\ &= E \left[ \left( \int_0^T \sigma_2(t) dW_2(t) \right)^2 \right] - 0^2 = \int_0^T \sigma_2(t)^2 dt \end{aligned}$$

where we have used Ito's isometry in the last step.

## Third Choice: Market models. Guided Tour

We thus have

$$I(T) := \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2^2(t) dt \sim$$

$$\sim m + V\mathcal{N}(0, 1), \quad m = -\frac{1}{2} \int_0^T \sigma_2(t)^2 dt, \quad V^2 = \int_0^T \sigma_2(t)^2 dt$$

Recall that we have

$$F_2(T) = F_2(0) \exp(I(T)) = F_2(0) e^{m+V\mathcal{N}(0,1)}$$

Compute now the option price

$$E^{Q^2}[(F_2(T_1) - X)^+] = E^{Q^2}[(F_2(0)e^{m+V\mathcal{N}(0,1)} - X)^+]$$

$$= \int_{-\infty}^{+\infty} (F_2(0)e^{m+Vy} - X)^+ p_{\mathcal{N}(0,1)}(y) dy = \dots$$

Note that  $F_2(0) \exp(m + Vy) - X > 0$  if and only if

$$y > \frac{-\ln\left(\frac{F_2(0)}{X}\right) - m}{V} =: \bar{y}$$

## Third Choice: Market models. Guided Tour

so that

$$\begin{aligned}
 \dots &= \int_{\bar{y}}^{+\infty} (F_2(0) \exp(m + Vy) - X) p_{\mathcal{N}(0,1)}(y) dy = \\
 &= F_2(0) \int_{\bar{y}}^{+\infty} e^{m+Vy} p_{\mathcal{N}(0,1)}(y) dy - X \int_{\bar{y}}^{+\infty} p_{\mathcal{N}(0,1)}(y) dy \\
 &= F_2(0) \frac{1}{\sqrt{2\pi}} \int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}y^2 + Vy + m} dy - X(1 - \Phi(\bar{y})) \\
 &= F_2(0) \frac{1}{\sqrt{2\pi}} \int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}(y-V)^2 + m - \frac{1}{2}V^2} dy - X(1 - \Phi(\bar{y})) = \\
 &= F_2(0) e^{m - \frac{1}{2}V^2} \frac{1}{\sqrt{2\pi}} \int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}(y-V)^2} dy - X(1 - \Phi(\bar{y})) = \\
 &= F_2(0) e^{m - \frac{1}{2}V^2} \frac{1}{\sqrt{2\pi}} \int_{\bar{y}-V}^{+\infty} e^{-\frac{1}{2}z^2} dz - X(1 - \Phi(\bar{y})) = \\
 &= F_2(0) e^{m - \frac{1}{2}V^2} (1 - \Phi(\bar{y} - V)) - X(1 - \Phi(\bar{y})) = \\
 &= F_2(0) e^{m - \frac{1}{2}V^2} \Phi(-\bar{y} + V) - X\Phi(-\bar{y}) = \\
 &= F_2(0)\Phi(d_1) - X\Phi(d_2), \quad d_{1,2} = \frac{\ln \frac{F_2(0)}{X} \pm \frac{1}{2} \int_0^{T_1} \sigma_2^2(t) dt}{\sqrt{\int_0^{T_1} \sigma_2^2(t) dt}}
 \end{aligned}$$

## Third Choice: Market models. Guided Tour

$$\text{Cpl}(0, T_1, T_2, X) = P(0, T_2)\tau[F_2(0)\Phi(d_1) - X\Phi(d_2)],$$

$$d_{1,2} = \frac{\ln \frac{F_2(0)}{X} \pm \frac{1}{2} \int_0^{T_1} \sigma_2^2(t) dt}{\sqrt{\int_0^{T_1} \sigma_2^2(t) dt}}$$

**This is exactly the classic market Black's formula** for the  $T_1 - T_2$  caplet. The term in squared brackets can be also written as

$$= F_2(0)\Phi(d_1) - X\Phi(d_2), \quad d_{1,2} = \frac{\ln \frac{F_2(0)}{X} \pm \frac{1}{2}T_1 v_1(T_1)^2}{\sqrt{T_1} v_1(T_1)}$$

where  $v_1(T_1)$  is the time-averaged quadratic volatility

$$v_1(T_1)^2 = \frac{1}{T_1} \int_0^{T_1} \sigma_2(t)^2 dt.$$

Notice that in case  $\sigma_2(t) = \sigma_2$  is constant we have

$$v_1(T_1) = \sigma_2.$$

## Third Choice: Market models. Guided Tour

Summing up: take

$$dF(t; T_1, T_2) = \sigma_2 F(t; T_1, T_2) dW_2(t), \quad \text{mkt } F(0; T_1, T_2)$$

The current zero-curve  $T \mapsto L(0, T)$  is calibrated through **the initial market**  $F(0; T, S)$ 's. This dynamics in **under the numeraire**  $P(\cdot, T_2)$  (measure  $Q^2$ ), where  $W_2$  is a Brownian motion. We wish to compute

$$E \left[ \frac{B(0)}{B(T_2)} \tau (F(T_1; T_1, T_2) - X)^+ \right]$$

We obtain from the change of numeraire and under  $Q^2$ , assuming **lognormality of F**:

$$\begin{aligned} \text{Cpl}(0, T_1, T_2, X) &:= P(0, T_2) \tau E (F(T_1; T_1, T_2) - X)^+ \\ &= P(0, T_2) \tau [F(0; T_1, T_2) \Phi(d_1(X, F(0; T_1, T_2), \sigma_2 \sqrt{T_1})) \\ &\quad - X \Phi(d_2(X, F(0; T_1, T_2), \sigma_2 \sqrt{T_1}))], \\ d_{1,2}(X, F, u) &= \frac{\ln(F/X) \pm u^2/2}{u}, \end{aligned}$$

This is the Black formula used in the market to convert Cpl prices in volatilities  $\sigma$  and vice-versa. This dynamical model is thus compatible with Black's market formula. The key property is **lognormality** of  $F$  when taking the expectation.

## Third Choice: Market models. Guided Tour

The example just introduced is a simple case of what is known as “lognormal forward-LIBOR model”. It is known also as Brace-Gatarek-Musiela (1997) model, from the name of the authors of one of the first papers where it was introduced rigorously. This model was also introduced earlier by Miltersen, Sandmann and Sondermann (1997). Jamshidian (1997) also contributed significantly to its development. However, a common terminology is now emerging and the model is generally known as “LIBOR Market Model”. We will stick to “Lognormal Forward-LIBOR Model” (LFM), since this is more informative on the properties of the model: Modeling forward LIBOR rates through a lognormal distribution (under the relevant probability measures).

Question: Can this model be obtained as a special **short rate model**?

Is there a choice for the equation of  $r$  that is consistent with the above market formula, or with the lognormal distribution of  $F$ 's?

## Third Choice: Market models. Guided Tour

Is there a choice for the equation of  $r$  that is consistent with the above market formula, or with the lognormal distribution of  $F$ 's?

Again to fix ideas, let us choose a specific short-rate model and assume we are using the Vasicek model. The parameters  $k, \theta, \sigma, r_0$  are denoted by  $\alpha$ .

$$r_t = x_t, \quad dx_t = k(\theta - x_t)dt + \sigma dW_t.$$

Such model allows for an analytical formula for forward LIBOR rates  $F$ ,

$$F(t; T_1, T_2) = F^{VAS}(t; T_1, T_2; x_t, \alpha).$$

At this point one can try and price a caplet. To this end, one can compute the risk-neutral expectation

$$E \left[ \frac{B(0)}{B(T_2)} \tau(F^{VAS}(T_1; T_1, T_2, x_{T_1}, \alpha) - X)^+ \right].$$

This too turns out to be feasible, and leads to a function

$$U_C^{VAS}(0, T_1, T_2, X, \alpha).$$

## Third Choice: Market models. Guided Tour

Question: Is there a short-rate model compatible with the Market model? For VASICEK  $dx_t = k(\theta - x_t)dt + \sigma dW_t$ , rewritten under  $Q^2$ , we have

$$dF^{VAS}(t; T_1, T_2; x_t, \alpha) = \frac{\partial F^{VAS}}{\partial [t, x]} d[t \ x_t]' + \frac{1}{2} \frac{\partial^2 F^{VAS}}{\partial x^2} (dx_t)^2,$$

VS Lognormal  $dF(t; T_1, T_2) = vF(t; T_1, T_2)dW_2(t)$ .

$F^{VAS}$  is not lognormal, nor are  $F$ 's associated to other known short rate models. So no known short rate model is consistent with the market formula. Short rate models are calibrated through their particular formulas for caplets, but these formulas are not Black's market formula (although some are close).

When Hull and White (extended VASICEK) is calibrated to caplets one has the values of  $k, \theta, \sigma, x_0$  consistent with caplet prices, but these parameters don't have an **immediate intuitive meaning** for traders, who don't know **how to relate them to Black's market formula**. On the contrary, the parameter  $\sigma_2$  in the mkt model **has an immediate meaning as the Black caplet volatility of the market**. **There is an immediate link between model parameters and market quotes. Language is important.**

## Third Choice: Market models. Guided Tour

When dealing with several caplets involving different forward rates,

$$F_2(t) = F(t; T_1, T_2), \quad F_3(t) = F(t; T_2, T_3), \dots, \quad F_\beta(t) := F(t; T_{\beta-1}, T_\beta),$$

or with swaptions, different structures of instantaneous volatilities can be employed. One can select a different  $\sigma$  for each forward rate by assuming each forward rate to have a constant instantaneous volatility. Alternatively, one can select piecewise-constant instantaneous volatilities for each forward rate. Moreover, different forward rates can be modeled as each having different random sources  $Z$  that are **instantaneously correlated**. Modeling correlation is necessary for pricing payoffs depending on more than a single rate at a given time, such as swaptions.

Dynamics of  $F_k(t) = F(t, T_{k-1}, T_k)$  under  $Q^k$  (numeraire  $P(\cdot, T_k)$ ) is  $dF_k(t) = \sigma_k(t) F_k dZ_k(t)$ , lognormal distrib. (we have seen the example  $k = 2$  above). Dynamics under  $Q^i \neq Q^k$  for  $i < k$  and  $i > k$  is more involved and does not lead to a known distribution of  $F_k$  under such measures.

## Third Choice: Market models. Guided Tour

Dynamics of  $F_k(t) = F(t, T_{k-1}, T_k)$  under  $Q^k$  (numeraire  $P(\cdot, T_k)$ ) is  $dF_k(t) = \sigma_k(t) F_k dZ_k(t)$ , lognormal distrib.

The LIBOR market model is calibrated to caplets **automatically** through integrals of the squared deterministic functions  $\sigma_k(t)$ .

For example, if one takes constant  $\sigma_k(t) = \sigma_k$  (constant), then  $\sigma_k$  is the market caplet volatility for the caplet resetting at  $T_{k-1}$  and paying at  $T_k$ .

No effort or complicated nonlinear inversion / minimization is involved to solve the “reverse engineering” problem

$$\text{MarketCplPrice}(0, T_1, T_2, X_2) = \text{LIBORModelCplPrice}(\sigma_2?);$$

$$\text{MarketCplPrice}(0, T_2, T_3, X_3) = \text{LIBORModelCplPrice}(\sigma_3?);$$

$$\text{MarketCplPrice}(0, T_3, T_4, X_4) = \text{LIBORModelCplPrice}(\sigma_4?);$$

....

Whereas it is complicated to solve

$$\text{MarketCplPrice}(0, T_1, T_2, X_2) = \text{VasicekModelCplPrice}(k?, \theta?, \sigma?);$$

$$\text{MarketCplPrice}(0, T_2, T_3, X_3) = \text{VasicekModelCplPrice}(k?, \theta?, \sigma?);$$

$$\text{MarketCplPrice}(0, T_3, T_4, X_4) = \text{VasicekModelCplPrice}(k?, \theta?, \sigma?);$$

....

Swaptions can be calibrated through some algebraic formulas under some good approximations, and the swaptions market formula is almost compatible with the model.

## Third Choice: Market models. Guided Tour

The LIBOR market model for  $F$ 's allows for:

- immediate and intuitive calibration of caplets (better than any short rate model)
- easy calibration to swaptions through algebraic approximation (again better than most short rate models)
- can virtually calibrate a high number of market products exactly or with a precision impossible to short rate models;
- clear correlation parameters, since these are instantaneous correlations of market forward rates;
- Powerful diagnostics: can check **future** volatility and terminal correlation structures (Diagnostics impossible with most short rate models);
- Can be used for monte carlo simulation;
- High dimensionality (many  $F$  are evolving jointly).
- Unknown joint distribution of the  $F$ 's (although each is lognormal under its canonical measure)
- Difficult to use with partial differential equations or lattices/trees, but recent Monte Carlo approaches such as Least Square Monte Carlo make trees and PDE's less necessary.

**End of the guided tour to the LIBOR model**  
**Now we begin the detailed presentation.**

## Giving rigor to Black's formulas: The LFM market model in general

End of the guided tour to the LIBOR model  
Now we begin the detailed presentation.

Recall measure  $Q^U$  associated with numeraire  $U$

(Risk-neutral measure  $Q = Q^B$ ).

FACT 1:  $A/U$ , with  $A$  a tradable asset, is a  $Q^U$ -martingale

Caps: Rigorous derivation of Black's formula.

Take  $U = P(\cdot, T_i)$ ,  $Q^U = Q^i$ . Since

$$F(t; T_{i-1}, T_i) = (1/\tau_i)(P(t, T_{i-1}) - P(t, T_i))/P(t, T_i),$$

$F(t; T_{i-1}, T_i) =: F_i(t)$  is a  $Q^i$ -martingale. Take

$$dF_i(t) = \sigma_i(t)F_i(t)dZ_i(t), \quad Q^i, \quad t \leq T_{i-1}.$$

This is the **Lognormal Forward-Libor Model (LFM)**. Consider the discounted  $T_{k-1}$ -caplet

$$(F_k(T_{k-1}) - K)^+ B(0)/B(T_k)$$

## The LFM model dynamics in general

$$\mathbf{LFM:} \quad dF_k(t) = \sigma_k(t)F_k(t)dZ_k(t), \quad Q^k, \quad t \leq T_{k-1}.$$

The price at the time 0 of the single caplet is (use FACT 2)

$$\begin{aligned} & \boxed{B(0)} E^{\boxed{Q^B}} \left[ \frac{(F_k(T_{k-1}) - K)^+}{\boxed{B(T_k)}} \right] = \\ & = \boxed{P(0, T_k)} E^{\boxed{Q^k}} \left[ \frac{(F_k(T_{k-1}) - K)^+}{\boxed{P(T_k, T_k)}} \right] = \dots \\ & = P(0, T_k) \text{ B\&S}(F_k(0), K, v_{T_{k-1}-\text{caplet}} \sqrt{T_{k-1}}) \\ & \quad v_{T_{k-1}-\text{caplet}}^2 = \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_k(t)^2 dt \end{aligned}$$

The dynamics of  $F_k$  is easy under  $Q^k$ . But if we price a product depending on several forward rates at the same time, we need to fix a pricing measure, say  $Q^i$ , and model all rates  $F_k$  under this same measure  $Q^i$ .

In this case we are lucky when  $k = i$ , since things are easy, but we are in troubles when  $i < k$  or  $i > k$ , since the dynamics of  $F_k$  under  $Q^i$  (rather than  $Q^k$ ) becomes difficult. We are going to derive it now using the change of numeraire toolkit.

## The LFM model dynamics in general

### Dynamics of $F_k$ under $Q^i$ .

Consider the forward rate  $F_k(t) = F(t, T_{k-1}, T_k)$  and suppose we wish to derive its dynamics first under the  $T_i$ -forward measure  $Q^i$  with  $i < k$ . We know that the dynamics under the  $T_k$ -forward measure  $Q^k$  has null drift. From this dynamics, we propose to recover the dynamics under  $Q^i$ . Let us apply the change of numeraire toolkit. The change of numeraire toolkit provides the formula relating Brownian shocks under numeraire 2 (say  $U$ ) given shocks under Numeraire 1 (say  $S$ ). See for example Formula (2.13) in Brigo and Mercurio (2001), Chapter 2. We can write

$$dZ_t^S = dZ_t^U - \rho \left( \frac{DC(S)}{S_t} - \frac{DC(U)}{U_t} \right)' dt$$

where we abbreviate “Vector Diffusion Coefficient” by “DC”.

## The LFM model dynamics in general

DC is actually a sort of linear operator for diffusion processes that works as follows.  $DC(X_t)$  is the row vector  $\mathbf{v}$  in

$$dX_t = (\dots)dt + \mathbf{v} dZ_t$$

for diffusion processes  $X$  with  $Z$  column vector Brownian motion common to all relevant diffusion processes. This is to say that if for example  $dF_1 = \sigma_1 F_1 dZ_1^1$ , then

$$DC(F_1) = [\sigma_1 F_1, 0, 0, \dots, 0] = \sigma_1 F_1 e_1.$$

The correlation matrix  $\rho$  is the instantaneous correlation among all shocks (the same under any measure):

$$dZ_i dZ_j = \rho_{i,j} dt$$

The toolkit

$$dZ_t^S = dZ_t^U - \rho \left( \frac{DC(S)}{S_t} - \frac{DC(U)}{U_t} \right)' dt$$

can also be written as

$$dZ_t^S = dZ_t^U - \rho (DC(\ln(S/U)))' dt$$

## The LFM model dynamics in general

$$dZ_t^S = dZ_t^U - \rho (\text{DC}(\ln(S/U)))' dt$$

This alternative toolkit expression (which we shall use) is obtained by noticing that

$$\begin{aligned} \frac{\text{DC}(S)}{S_t} - \frac{\text{DC}(U)}{U_t} &= \text{DC}(\ln(S)) - \text{DC}(\ln(U)) \\ &= \text{DC}(\ln(S) - \ln(U)) = \text{DC}(\ln(S/U)) \end{aligned}$$

Let us apply the toolkit:  $S = P(\cdot, T_k)$  and  $U = P(\cdot, T_i)$

$$dZ_t^k = dZ_t^i - \rho \text{DC}(\ln(P(\cdot, T_k)/P(\cdot, T_i)))' dt$$

Now notice that

$$\begin{aligned} \ln \frac{P(t, T_k)}{P(t, T_i)} &= \ln \left( \frac{P(t, T_k)}{P(t, T_{k-1})} \frac{P(t, T_{k-1})}{P(t, T_{k-2})} \cdots \frac{P(t, T_{i+1})}{P(t, T_i)} \right) = \\ &= \ln \left( \frac{1}{1 + \tau_k F_k(t)} \cdot \frac{1}{1 + \tau_{k-1} F_{k-1}(t)} \cdots \frac{1}{1 + \tau_{i+1} F_{i+1}(t)} \right) = \\ &= \ln \left( 1 / \left[ \prod_{j=i+1}^k (1 + \tau_j F_j(t)) \right] \right) = - \sum_{j=i+1}^k \ln (1 + \tau_j F_j(t)) \end{aligned}$$

## The LFM model dynamics in general

$$\ln \frac{P(t, T_k)}{P(t, T_i)} = - \sum_{j=i+1}^k \ln (1 + \tau_j F_j(t))$$

so that from linearity

$$\begin{aligned} \text{DC} \ln \frac{P(t, T_k)}{P(t, T_i)} &= - \sum_{j=i+1}^k \text{DC} \ln (1 + \tau_j F_j(t)) \\ &= - \sum_{j=i+1}^k \frac{\text{DC}(1 + \tau_j F_j(t))}{1 + \tau_j F_j(t)} = - \sum_{j=i+1}^k \tau_j \frac{\text{DC}(F_j(t))}{1 + \tau_j F_j(t)} = \\ &= - \sum_{j=i+1}^k \tau_j \frac{\sigma_j(t) F_j(t) e_j}{1 + \tau_j F_j(t)} \end{aligned}$$

where  $e_j$  is a zero row vector except in the  $j$ -th position, where we have 1 (vector diffusion coefficient for  $dF_j$  is  $\sigma_j F_j e_j$ ). Recalling

$$dZ_t^k = dZ_t^i - \rho \text{DC}(\ln(P(\cdot, T_k)/P(\cdot, T_i)))' dt$$

we may now write

$$dZ_t^k = dZ_t^i + \rho \sum_{j=i+1}^k \tau_j \frac{\sigma_j(t) F_j(t) e_j'}{1 + \tau_j F_j(t)} dt$$

## The LFM model dynamics in general

$$dZ_t^k = dZ_t^i + \rho \sum_{j=i+1}^k \tau_j \frac{\sigma_j(t) F_j(t) e_j'}{1 + \tau_j F_j(t)} dt$$

Pre-multiply both sides by  $e_k$ . We obtain

$$\begin{aligned} dZ_k^k &= dZ_k^i + [\rho_{k,1} \ \rho_{k,2} \dots \rho_{k,n}] \sum_{j=i+1}^k \tau_j \frac{\sigma_j(t) F_j(t) e_j'}{1 + \tau_j F_j(t)} dt \\ &= dZ_k^i + \sum_{j=i+1}^k \tau_j \frac{\sigma_j(t) F_j(t) \rho_{k,j}}{1 + \tau_j F_j(t)} dt \end{aligned}$$

Substitute this in our usual equation  $dF_k = \sigma_k F_k dZ_k^k$  to obtain

$$dF_k = \sigma_k F_k \left( dZ_k^i + \sum_{j=i+1}^k \tau_j \frac{\sigma_j(t) F_j(t) \rho_{k,j}}{1 + \tau_j F_j(t)} dt \right)$$

that is finally the equation showing the dynamics of a forward rate with maturity  $k$  under the forward measure with maturity  $i$  when  $i < k$ . The case  $i > k$  is analogous.

## The LFM model dynamics in general

$$dF_k(t) = \mu_i^k(t, F(t))\sigma_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t),$$

$$dF_k(t) = \sigma_k(t)F_k(t)dZ_k^k(t)$$

$$dF_k(t) = -\mu_k^i(t, F(t))\sigma_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t),$$

for  $i < k$ ,  $i = k$  and  $i > k$  respectively, where we have set

$$\mu_l^m = \sum_{j=l+1}^m \tau_j \frac{\sigma_j(t)F_j(t)\rho_{m,j}}{1 + \tau_j F_j(t)}$$

As for existence and uniqueness of the solution, the case  $i = k$  is trivial. In the case  $i < k$ , use Ito's formula:

$$d \ln F_k(t) = \sigma_k(t) \sum_{j=i+1}^k \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1 + \tau_j F_j(t)} dt - \frac{\sigma_k(t)^2}{2} dt + \sigma_k(t)dZ_k(t).$$

The diffusion coefficient is deterministic and bounded. Moreover, since  $0 < \tau_j F_j(t)/(1 + \tau_j F_j(t)) < 1$ , also the drift is bounded, besides being smooth in the  $F$ 's (that are positive). This ensures existence and uniqueness of a strong solution for the above SDE. The case  $i > k$  is analogous.

## LIBOR model under the Spot Measure

It may happen that in simulating forward rates  $F_k$  under numeraires  $Q^i$  with  $i$  much larger or smaller than  $k$ , the effect of the discretization procedure worsens the approximation with respect to cases where  $i$  is closer to  $k$ .

A remedy to situations where we may need to simulate  $F_k$  very far away from the numeraire  $Q^i$  is to adopt the spot measure.

Consider a discretely rebalanced bank-account numeraire as an alternative to the continuously rebalanced bank account  $B(t)$  (whose value, at any time  $t$ , changes according to  $dB(t) = r_t B(t) dt$ ). We introduce a bank account that is rebalanced only on the times in our discrete-tenor structure. To this end, consider the numeraire asset

$$B_d(t) = \frac{P(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P(T_{j-1}, T_j)} = \prod_{j=0}^{\beta(t)-1} (1 + \tau_j F_j(T_{j-1})) P(t, T_{\beta(t)-1}).$$

Here in general  $T_{\beta(u)-2} < u \leq T_{\beta(u)-1}$ .

## LIBOR model under the Spot Measure

$$B_d(t) = \frac{P(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P(T_{j-1}, T_j)} = \prod_{j=0}^{\beta(t)-1} (1 + \tau_j F_j(T_{j-1})) P(t, T_{\beta(t)-1}).$$

The interpretation of  $B_d(t)$  is that of the value at time  $t$  of a portfolio defined as follows. The portfolio starts with one unit of currency at  $t = 0$ , exactly as in the continuous-bank-account case ( $B(0)=1$ ), but this unit amount is now invested in a quantity  $X_0$  of  $T_0$  zero-coupon bonds. Such  $X_0$  is readily found by noticing that, since we invested one unit of currency, the present value of the bonds needs to be one, so that  $X_0 P(0, T_0) = 1$ , and hence  $X_0 = 1/P(0, T_0)$ . At  $T_0$ , we cash the bonds payoff  $X_0$  and invest it in a quantity  $X_1 = X_0/P(T_0, T_1) = 1/(P(0, T_0)P(T_0, T_1))$  of  $T_1$  zero-coupon bonds. We continue this procedure until we reach the last  $T_{\beta(t)-2}$  preceding the current time  $t$ , where we invest  $X_{\beta(t)-1} = 1/\prod_{j=1}^{\beta(t)-1} P(T_{j-1}, T_j)$  in  $T_{\beta(t)-1}$  zero-coupon bonds. The present value at the current time  $t$  of this investment is  $X_{\beta(t)-1}P(t, T_{\beta(t)-1})$ , i.e. our  $B_d(t)$  above. Thus,  $B_d(t)$  is obtained by starting from one unit of currency and reinvesting at each tenor date in zero-coupon bonds for the next tenor. This gives a discrete-tenor counterpart of  $B$ , and the subscript “ $d$ ” in  $B_d$  stands for “discrete”.  $B_d$  is also called **discretely rebalanced bank account**

## LIBOR model under the Spot Measure

Now choose  $B_d$  as numeraire and apply the change-of-numeraire technique starting from the dynamics  $dF_k = \sigma_k F_k dZ_k$  under  $Q^k$ , to obtain the dynamics under  $B_d$ . Calculations are analogous to those given for the  $Q^i$  case.

The measure  $Q^d$  associated with  $B_d$  is called *spot LIBOR measure*. We then have the following **(Spot-LIBOR-measure dynamics in the LFM)**

$$dF_k(t) = \sigma_k(t) F_k(t) \sum_{j=\beta(t)}^k \frac{\tau_j \rho_{j,k} \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t) F_k(t) dZ_k^d(t).$$

Both the spot-measure dynamics and the risk-neutral dynamics admit no known transition densities, so that the related equations need to be discretized in order to perform simulations.

## LIBOR model under the Spot Measure: Benefits

Assume we are in need to value a payoff involving rates  $F_1, \dots, F_{10}$  from time 0 to time  $T_9$ .

Consider two possible measures under which we can do pricing.

First  $Q^{10}$ . Under this measure, consider each rate  $F_j$  in each interval with the number of terms in the drift summation of each rate shown between square brackets:

$$\begin{aligned} 0 \div T_0 : & F_1[9], F_2[8], F_3[7], \dots, F_9[1], F_{10}[0] \\ T_0 \div T_1 : & F_2[8], F_3[7], \dots, F_9[1], F_{10}[0] \\ T_1 \div T_2 : & F_3[7], \dots, F_9[1], F_{10}[0] \end{aligned}$$

etc. Notice that if we discretize some rates will be more biased than others. Instead, with the spot LIBOR measure

$$\begin{aligned} 0 \div T_0 : & F_1[1], F_2[2], F_3[3], \dots, F_9[9], F_{10}[10] \\ T_0 \div T_1 : & F_2[1], F_3[2], \dots, F_9[8], F_{10}[9] \\ T_1 \div T_2 : & F_3[1], \dots, F_9[7], F_{10}[8] \end{aligned}$$

etc. Now the bias, if any, is more distributed.